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Solving optimal control problems of a linear oscillator with the differential equation

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Abstract

A computational strategy for solving optimal control problems (OCPs) is displayed. This technique is based on the Bezier curve strategy. Thus, an optimal control problem (OCP) with pantograph delays converts to an optimization problem, which can be solved easily. Numerical examples are given to exhibit the applicability and efficiency of the technique.

Keywords: Bezier curve, Boubaker polynomials, Optimal control problems
Mathematics Subject Classification [2010]: 65K10, 26A33

1 Introduction

OCPs have an important role in some areas including engineering economics and finance. A computational strategy for solving OCP is developed by Wu, et al. [6] which is obtained by a switched dynamical system with time delay. Kharatishidi [3] approached this problem by extending the Pontryagin's maximum principle to time delay systems (TDS). The actual solution involves a two-point boundary value problem in which advances and delays are stated. In addition, this solution does not yield a feedback controller. OCP with time delay has been considered by Oguztoreli [5] who achieving several findings concerning bang-bang controls which are parallel to those of LaSalle [4] for non delay systems. For a time invariant system with an infinite upper limit in the performance measure, An optimal regulator for a linear system with multiple states, input delays and a quadratic criterion is presented in some papers. The optimal regulator equations were achieving by reducing original problem to the linear quadratic regulator design for a system without delays. In this paper, we will solve OCP by Bezier curve.
The outline of this paper is as follows: In Section 2, Bezier curve technique is introduced. Also, a remark is stated. Some examples are given in Section 3. Section 4 is dedicated the conclusion.

2 Bezier curve technique

Our system is utilizing Bezier curves to approximate the solutions $x_i(t)$ and $u(t)$ where $x_i(t)$ and $u(t)$ are given below. Define the Bezier polynomials of degree n over the interval $[t_0, t_f]$ as follows:

$$x_i(t) = \sum_{r=0}^{n} a_r^i B_{r,n}(\frac{t-t_0}{h}), \tag{1}$$

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$$u(t) = \sum_{r=0}^{n} b_r B_{r,n}(\frac{t-t_0}{h}) \tag{2}$$

where $h = t_f - t_0$, and

$$B_{r,n}(\frac{t-t_0}{h}) = \binom{n}{r} \frac{1}{h^n} (t_f - t)^{n-r} (t - t_0)^r,$$

is the Bernstein polynomial of degree n over the interval $[t_0, t_f]$, a_r^i and b_r are the control points. By substituting $x_i(t)$ and $u(t)$ in the optimal control of a direct oscillator with the differential equation, one may solve these problems by Maple 15.
Ghomanjani et al. [1] demonstrated the convergence of this technique when $n \to \infty$.

Remark 2.1. The optimal control of a linear oscillator with the differential equation will be considered

$$u(t) \;=\; \ddot{x}(t) + w^2 x(t), \quad t \in [-T, 0] \tag{3}$$

when T is specified. Eq. (3) is equivalent to the following state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -w^2 x_1(t) + u(t), \\ x_1(-T) &= x_0, \quad x_2(-T) = \dot{x}_0, \\ x_1(0) &= 0, \quad x_2(0) = 0. \end{aligned}$$

with the following cost function

$$J \;=\; \frac{1}{2} \int_{-T}^{0} u^2(t)\, dt,$$

when it has the following exact solution (see [2]):

$$\begin{aligned} &x_1(t) = \frac{1}{2w^2}[Awt\sin(wt) + B(\sin(wt) - wt\cos(wt))], \\ &x_2(t) = \frac{1}{2w}[A(wt\sin(wt) + wt\cos(wt)) + Bwt\sin(wt)], \\ &u(t) = A\cos(wt) + B\sin(wt), \\ &J = \frac{1}{8w}[2wT(A^2 + B^2) + (A^2 - B^2)\sin(2wT) - 4AB\sin^2(wT)], \end{aligned}$$

where

$$\begin{aligned} A &= \frac{2w[x_0 w^2 T\sin(wT) - \dot{x}_0(wT\cos(wT) - \sin(wT))]}{w^2T^2 - \sin^2(wT)} \\ B &= \frac{2w^2[\dot{x}_0 T\sin(wT) + x_0(wT\cos(wT) + \sin(wT))]}{w^2T^2 - \sin^2(wT)} \end{aligned}$$

3 Numerical applications

Now, some numerical examples are explained.

Example 3.1. Consider the following problem

$$\min J = \frac{1}{2}\int_0^2 u^2(t)\ dt,$$
$$s.t.\ \ u(t) = \ddot{x}(t) + \dot{x}(t),$$
$$x(0) = 0,\ x(2) = 5,\ \dot{x}(0) = 0,\ \dot{x}(2) = 2,$$

when the exact solutions for $x(t)$ and $u(t)$ are

$$\begin{aligned} x(t) &= -6.103 + 7.289t + 6.696e^{-t} - 0.593e^{t}, \\ u(t) &= 7.289 - 1.186e^{t}, \end{aligned}$$

utilizing the proposed strategy with $n = 10$, one may have $J = 16.75072576$ where the exact value is $J_{exact} = 16.74543860$, and

$$\begin{aligned} x_{approx}(t) &= -1.215249320t^3 - 0.6028389216 * 10^{-1}t^5 + 0.7871472800 * 10^{-2}t^6 \\ &- 0.9888591200 * 10^{-3}t^7 - 0.5385276000 * 10^{-4}t^8 + 0.3124650000 * 10^{-4}t^9 \\ &- 0.4281570000 * 10^{-5}t^10 + .2541508128t^4 + 3.052409559t^2 \\ &- 1.659151229 * 10^{-307} \\ u_{approx}(t) &= -1.186676818t - .1890746400t^3 + 0.5697011680 * 10^{-2}t^5 \\ &- 0.9937861840 * 10^{-2}t^6 + 0.1818916840 * 10^{-2}t^7 - 0.1041147500 * 10^{-3}t^8 \\ &- 0.4281581000 * 10^{-4}t^9 - 0.6527546688 * 10^{-1}t^4 - .5959381133t^2 \\ &+ 6.104819121 \end{aligned}$$

The graphs of approximated and exact solution $x(t)$ and $u(t)$ are plotted respectively in Figs. 1, 2.

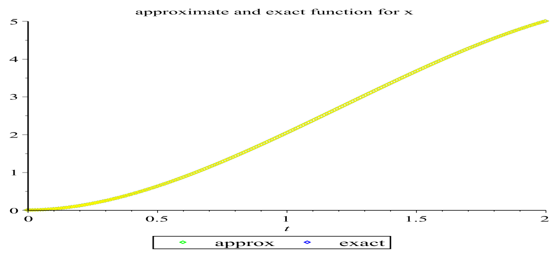

Figure 1: The graph of approximated and exact solution for Example 3.1

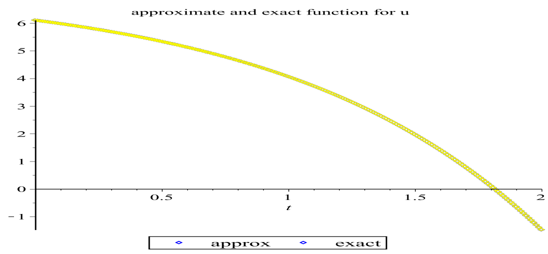

Figure 2: The graph of approximated and exact solution for Example 3.1

Example 3.2. Consider the following problem

$$\min J = \frac{1}{2}\int_{-2}^{0} u^2(t)\ dt,$$
$$s.t.\ \ u(t) = \ddot{x}(t) + x(t),$$
$$x(-2) = 0.5,\ x(0) = 0,\ \dot{x}(-2) = -0.5,\ \dot{x}(0) = 0,$$

utilizing the proposed technique with $n = 10$, one may have $J = 0.1848585418$ where the exact value is $J_{exact} = 0.1848585422$ and

$$\begin{aligned} x_{approx}(t) &= 7.062378300 * 10^{-7} * t^{10} + 0.6874780960 * 10^{-5}t^8 + 0.1194692496 * 10^{-3}t^6 \\ &+ 0.9304003280 * 10^{-5}t^9 - 0.3102224362 * 10^{-3}t^7 + 0.9160017711 * 10^{-2}t^5 \\ &- 0.9147355857 * 10^{-1}t^3 - 0.2017277233 * 10^{-2}t^4 + 0.1213363708 * 10^{-1}t^2, \\ u_{approx}(t) &= 7.062378300 * 10^{-7}t^{10} + 0.7043560663 * 10^{-4}t^8 + 0.5044622766 * 10^{-3}t^6 \\ &- .5488413512t + 0.9304003280 * 10^{-5}t^9 + 0.3596649426 * 10^{-3}t^7 \\ &- 0.3869309236 * 10^{-2}t^5 + 0.9172680103 * 10^{-1}t^3 + 0.1566814558 * 10^{-2}t^4 \\ &- 0.1207368883 * 10^{-1}t^2 + 0.2426727416 * 10^{-1}. \end{aligned}$$

The graphs of approximated and exact solution $x(t)$ and $u(t)$ are plotted respectively in Figs. 3, 4.

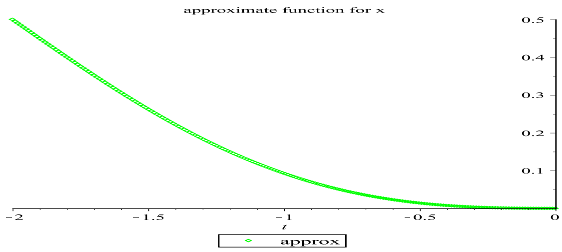

Figure 3: The graph of approximated solution for Example 3.2

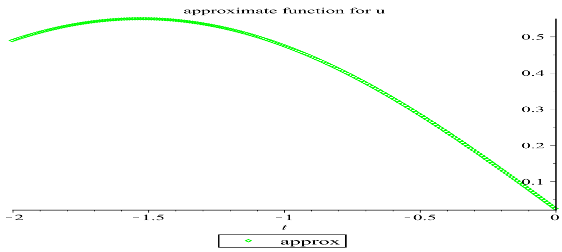

Figure 4: The graph of approximated solution for Example 3.2

4 Conclusion

This paper presents a numerical technique for solving non linear OCPs by Bezier curve technique. The efficiency of the method was obtained by some numerical examples.

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