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On the ring of continuous linear transformations of vector spaces

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Abstract

Let V be a real or complex locally convex vector space and $L_c(V)$ be the ring of continuous linear operators on V. In this talk, some results related to the characterization and properties of one-sided ideals of the ring $L_c(V)$ which have been obtained over the past decades are reviewed. Moreover, some outlines for the continuation of the subject under investigation is presented.

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1 Introduction

Throughout this talk, V denotes a real or complex locally convex vector space. The symbols $L_c(V)$ and $B(V)$ are used to the ring of continuous linear operators on V and the algebra of bounded linear operators on V, respectively. Moreover, the terms subspace and (linear) operator are, respectively, used to describe a closed subspace of a locally convex vector space V and a continuous linear operator on V. A subspace M is called invariant for a collection F in $L_c(V)$ if $T(M) \subseteq M$ for all $T \in F$. A collection F of linear operators $L_c(V)$ is called reducible if $F = \{0\}$ or it has a non-trivial invariant subspace. The collection F is irreducible if it is not reducible. Moreover, F is called simultaneously triangularizable or simply triangularizable if there exists a maximal chain of the subspaces of V each of which is invariant under the collection F. The (topological) dual of V, denoted by the symbol V^* is defined to $L_c(X, \mathbb{F})$, the set of all continuous linear functionals from V into $\mathbb{F}$ where, $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$. Let n be a positive integer and $\{x_i\}_{i=1}^n$ be a finite independent subset of V. Then, there exists an independent subset $\{f^i\}_{i=1}^n$ of continuous linear functionals on V satisfying $f_i(x_j) = \delta ij$, where $i, j = 1, 2, \ldots, n$ and δ denotes the Kronecker delta. Every such independent subset of V is called a dual independent subset with respect to $\{x_i\}_{i=1}^n$. Recall that the weak* topology of V is the weakest topology on V for which every $\hat{x} \in V^{**}$ is continuous, where $x \in V$ and $\hat{} : V \to V^{**}$ is the natural mapping from V into V^{**}. The space V under its weak* topology is a locally convex vector space. Let V and W be locally convex vector spaces (resp. normed linear spaces) over $\mathbb{F}$. For $T \in L_c(V, W)$, the set of all continuous linear operators from V into W (resp. $T \in B(V, W)$, the set of all bounded linear

*Speaker. Email address: najafizadeh@pnu.ac.ir operators from V into W), we use the symbol T^* to denote the (topological) adjoint of T which is defined as the restriction of the algebraic adjoint of T to W^*. In this talk some specific one-sided ideals of the ring $L_c(V)$ are presented. The main result of the talk is about a characterization of certain one-sided ideals of the ring $L_c(V)$ in terms of their rank-one idempotents. Moreover, it is shown that a one-sided ideal of the ring of continuous linear operators on a real or complex locally convex space is triangularizable exactly if it is generated by a rank-one idempotent. This is also equivalent to this that the rank of $TS - ST$ is less or equal to one for all T and S in such a one-sided ideal. Finally, a description of irreducible one-sided ideals of the ring $L_c(V)$ in terms of their images or co-images will be given. The results are reviewed from [4], however the research is in progress by the authors which have led to some results and have not been published yet. The improvements relevant to this subject have been done over the past decades by some authors such as [2–5].

2 Main results

Let D be a division ring, V and W be right vector spaces over D, and $L(V, W)$ be the set of all right linear transformations from V to W. If $V = W$, then we use the symbol $L(V)$ to denote $L(V, V)$. The set $L(V)$ forms a ring under the addition and composition of linear transformations. An important subset of $L(V, W)$ is the class of rank-one linear transformations. Every rank-one linear transformation is of the form $x \otimes f$ for some $x \in W$ and $f \in V^*$, where $(x \otimes f)(y) = xf(y)$ for all $y \in V$. For a subset S of V, we define $(S)^\perp = \{f \in V^* : f(S) = 0\}$ which is a weak* closed subspace of V^*. We begin with a lemma from [3].

Lemma 2.1. *Let V and W be two vector spaces over a division ring D and $C \subseteq L(V, W)$. Then $(\cap_{T\in C} \ker T)^\perp = \langle \cup_{T\in C} T^* W^* \rangle$. Moreover, we have $\langle \cup_{T\in C} TV \rangle^\perp = \cap_{T\in C} \ker T^*$.*

For the case of locally convex spaces, we have the following lemma ([4, Lemma 1.1]).

Lemma 2.2. *Let V and W be locally convex spaces and $C \subseteq L_c(V, W)$. Then $(\cap_{T\in C} \ker T)^\perp = \overline{\langle \bigcup_{T\in C} T^* W^* \rangle}^{w*}$. Moreover, $\langle \cup_{T\in C} TV \rangle^\perp = \cap_{T\in C} \ker T^*$, where $\overline{\langle \bigcup_{T\in C} T^* W^* \rangle}^{w*}$ denotes the closure of $\langle \cup_{T\in C} T^* W^* \rangle$ in the weak∗ topology of V^*.*

For a family $F \subseteq L(V)$, the image of the family F denoted by $im(F)$ is the subspace of V generated by $\{T(x) : T \in F, x \in V\}$. Moreover, the kernel of this family is denoted by $\ker(F)$ which is defined as $\cap_{T\in F} \ker(T)$. The co-image and co-kernel of the family F, denoted by $coim(F)$ and $coker(F)$ respectively, are defined as $V/\ker F$ and $V/im(F)$. The following theorem characterizes all right (resp. left) ideals in $L_c(V)$ whose images (resp. co-images) are finite-dimensional.

Theorem 2.3. *Let V be a locally convex vector space and I be a non-zero right ideal in $L_c(V)$. If the image of I is finite-dimensional, then there exist some $x_i \in V$ and $f_i \in V^*(1 \le i \le r)$ which are dual to each other such that $I = x_1 \otimes f_1 L_c(V) + \ldots + x_r \otimes f_r L_c(V) = AL_c(V)$, where $r = dim\, im(I)$. In the case that I is a non-zero left ideal in $L_c(V)$ such that the co-image of I is finite-dimensional, then there are $x_i \in V$ and $f_i \in V^*(1 \le i \le r)$ which are dual to each other such that $I = L_c(V)x_1 \otimes f_1 + \ldots + L_c(V)x_r \otimes f_r = L_c(V)A$, where $r = codim\, im(I)$. In both cases $A = x_1 \otimes f_1 + \ldots + x_r \otimes f_r$ is an idempotent in I. Therefore, every right (resp. left) ideal of $L_c(V)$ whose image (resp. co-image) is finite- dimensional is principal; in fact such a right (resp. left) ideal is generated by a finite-rank idempotent whose rank is equal to the dimension of the image (resp. co-image) of the right (resp. left) ideal.*

Proof. We consider to cases. In the first case, we assume that I is a right ideal with a finite-dimensional image in $L_c(V)$. Choose $A_i \in I$ and $y_i \in V$ for $i = 1, 2, \ldots, r$ such that $\{A_i y_i\}_{1\le i\le r}$

is a basis for $im(I)$. Set $x_i = A_i y_i$. Now, in view of [6, Corollary II.4.2.1], let $\{f_i\}_{1\le i\le r}$ be a dual subset with respect to $\{x_i\}_{1\le i\le r}$ so that $f_i(x_j) = \delta_{ij}$ for each $i, j = 1, \ldots, r$. We show that $I = x_1\otimes f_1 L_c(V)+\ldots+x_r\otimes f_r L_c(V)$. We have $x_i\otimes f_i = A_i(y_i\otimes f_i)$. Since $A_i \in I$ and I is a right ideal in $L_c(V)$, it follows that $x_i\otimes f_i \in I$ for each $i = 1, \ldots, r$, whence $x_1\otimes f_1 L_c(V)+\ldots+x_r\otimes f_r L_c(V) \subseteq I$. On the other hand, since $\{f_i\}_{1\le i\le r}$ is dual to the basis $\{x_i\}_{1\le i\le r}$ of $im(I)$, it is easily seen that $B = x_1 \otimes f_1 B + \ldots + x_r \otimes f_r B = (x_1 \otimes f_1 + \ldots + x_r \otimes f_r)B$ for all $B \in I$. So we have shown that $I = x_1 \otimes f_1 L_c(V) + \ldots + x_r \otimes f_r L_c(V) = AL_c(V)$, where $A \in I$ is the idempotent $x_1 \otimes f_1 + \ldots + x_r \otimes f_r \in L_c(V)$. In the second case, we suppose that I is a left ideal in $L_c(V)$ whose coimage is finite-dimensional. Let $r = dim(V/\ker I)$. Choose an element x_1 such that $x_1 \notin \ker I$ so that $A_1x_1 \neq 0$ for some $A_1 \in I$. Now by [1, Corollary 2.2.1], we obtain a continuous linear functional f such that $f(A_1x_1) = 1$. Then $E_1 = x_1 \otimes fA_1 = (x_1 \otimes f)A_1$ is a rank-one idempotent in I sending x_1 to $x1$, annihilating $\ker I$. Let $1 < m \le r$ be an integer. Assume that we have found linearly independent vectors $x_1 + \ker I, \ldots, x_{m-1} + \ker I$ in $V/\ker I$ and a family of rank-one idempotents $E_1, \ldots, E_{m-1}$ in I such that $E_i(\ker I) = 0$ and that $E_ix_j = \delta_{ij}x_j$ for $i, j = 1, \ldots, m-1$. Therefore, $F_m := E_1 + \ldots + E_{m-1}$ is an idempotent in I of rank $m-1$ having $\ker I$ in its kernel and the vectors $\{x_1, \ldots, x_{m-1}\}$ in its range. Now choose x_m in the kernel of F_m such that $x_1 + \ker I, \ldots, x_{m-1} + \ker I, x_m + \ker I$ are linearly independent in $V/\ker I$. Again, just as in the above by [1, Corollary 2.2.1], there exists a rank-one idempotent $C_m \in I$ sending x_m to itself, annihilating $\ker I$. Obviously, $E_m := C_m(I - F_m) = C_m - C_mF_m$ is a rank-one idempotent in I sending x_m to itself and including the range of F_m in its kernel. Since $V/\ker I$ has finite dimension, finite induction implies that there exist a basis $\{x_1 + \ker I, \ldots, x_r + \ker I\}$ of $V/\ker I$ and a family of rank-one idempotents $\{E_1, \ldots, E_r\}$ in I such that $E_i(\ker I) = 0$ and $E_ix_j = \delta_{ij}x_j$ for all $i, j = 1, \ldots, r$. Clearly, $E_i = x_if_i$ for some linear functional f_i, where $i = 1, \ldots, r$. Therefore, we have $f_i(\ker I) = 0$ and $f_i(x_j) = \delta_{ij}$ for all $i, j = 1, \ldots, r$. Now, since $x_i\otimes f_i \in I$ for all $i = 1, \ldots, r$, it follows that $L_c(V)x_1\otimes f_1+\ldots+L_c(V)x_r\otimes fr \subseteq I$. On the other hand, if $B \in I$ is arbitrary, as the set $\{x_1 + \ker I, \ldots, x_r + \ker I\}$ is a basis for $V/\ker I$, we easily see that $B = Bx_1 \otimes f_1 + \ldots + Bx_r \otimes f_r$, proving that $I \subseteq L_c(V)x_1 \otimes f_1 + \ldots + L_c(V)x_r \otimes f_r$. Hence, $I = L_c(V)x_1 \otimes f_1 + \ldots + L_c(V)x_r \otimes f_r = L_c(V)A$. □

Corollary 2.4. *Let V be a locally convex vector space and $T \in L_c(V)$ be an arbitrary continuous linear operator. Then the following are equivalent.*

1. *T has rank equal to r.*
2. *The right ideal of $L_c(V)$ generated by T is $TL_c(V) = x_1 \otimes f_1L_c(V) + \ldots + x_r \otimes f_rL_c(V)$, where $\{x_i\}_{1\le i\le r}$ is a basis for $im(T)$ and f_i's are dual to x_i's $(1 \le i \le r)$.*
3. *The left ideal of $L_c(V)$ generated by T is $L_c(V)T = L_c(V)x_1 \otimes f_1 + \ldots + L_c(V)x_r \otimes f_r$, where $\{x_i + \ker T\}_{1\le i\le r}$ is a basis for coim (T) and f_i's are dual to x_i's $(1 \le i \le r)$.*

The following theorem shows that a one-sided ideal of $L_c(V)$ containing a linear operator T whose rank is maximal among all elements of the ideal is generated by T.

Theorem 2.5. *Let V be a locally convex vector space and I be a non-zero right ideal in $L_c(V)$ containing a linear operator T whose rank $r \in \mathbb{N}$ is maximal among all elements of I. Then, there exist $x_i \in V$ and $f_i \in V^*(1 \le i \le r)$ which are dual to each other such that the right ideal of $L_c(V)$ generated by T is $I = TL_c(V) = x_1 \otimes f_1L_c(V) + \ldots + x_r \otimes f_rL_c(V)$. Moreover, each x_i can be chosen to be in the range of T. In particular, if V is finite-dimensional, then the existence of $T \in I$ having maximal rank is clear, and hence the above holds for all right ideals of $L_c(V)$.*

For the case of left ideals we get:

Theorem 2.6. *Let V be a locally convex vector space and I be a non-zero left ideal in $L_c(V)$ containing a linear operator V whose rank $r \in \mathbb{N}$ is maximal among all elements of I. Then, there exist some $x_i \in V$ and $f_i \in V^*(1 \leq i \leq r)$ which are dual to each other such that the left ideal of $L_c(V)$ generated by T is $I = L_c(V)T = L_c(V)x_1 \otimes f_1 + \ldots + L_c(V)x_r \otimes f_r$. Moreover, each x_i can be chosen to be in the complement of the kernel of T. In particular, if V is finite-dimensional, then the existence of $T \in I$ having maximal rank is clear, and hence the above holds for all left ideals of $L_c(V)$.*

The following two theorems characterize all one-sided ideals of $L_c(V)$ and $B(V)$ that are irreducible.

Theorem 2.7. *Let V be a locally convex vector space and I be a right ideal in $L_c(V)$. Then I is irreducible if and only if $\overline{imI} = V$ where $\overline{imI}$ denotes the closure of the image of I in V. Moreover, in the case that I is a left ideal, then I is irreducible exactly if $coim(I) = V$ which is equivalent to* $\ker(I) = 0$.

Theorem 2.8. *Let V be a normed linear space over $\mathbb{R}$ or $\mathbb{C}$ and I be a right ideal in $B(V)$. Then the following are equivalent.*

1. *The closure of I in the operator norm topology of $B(V)$, denoted by $\bar{I}$, includes all finite-rank operators in $B(V)$.*
2. *I is irreducible.*
3. *$\overline{Im(I)} = V$.*

Therefore, if V is finite-dimensional, then $I = B(V)$ exactly if I is irreducible if and only if $Im\, I = V$.

We observe that if V is a normed linear space and I a left ideal in $B(V)$, then using Lemma 2.2, we can show that the closure of I in the strong operator topology of $B(V)$ includes all finite-rank operators if and only if the left ideal I is irreducible if and only if $coim\ (I) = V$, equivalently, $\ker I = 0$.

The final result characterizes all one-sided ideals of $L_c(V)$ that are triangularizable.

Theorem 2.9. *Let V be an arbitrary locally convex vector space and I be a nonzero right or left ideal in $L_c(V)$. Then the following are equivalent:*

1. *I is triangularizable.*
2. *I is generated by a rank-one idempotent.*
3. *I consists of all operators of rank at most one.*
4. *The rank of $TS - ST$ is at most one for all S and T in I.*

In particular, a linear operator $T \in L_c(V)$ has rank one exactly if one of the one-sided ideals generated by T is triangularizable [2].

3 Conclusion

The main result of the talk is about a characterization of certain one-sided ideals of the ring $L_c(V)$ in terms of their rank-one idempotents. Moreover, it is shown that a one-sided ideal of the ring of continuous linear operators on a real or complex locally convex space is triangularizable exactly if it is generated by a rank-one idempotent. The research for the subject under investigation, in particular for some specific kinds of rings of linear transformations is in progress by the authors which have led to some results and have not been published yet.

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