A new majorization and preservers

Abbas Askarizadeh[1*]
[1]Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

Abstract

In this manuscript by using the Hadamard product on the set of all m-by-n real matrices, we define a type of majorization and study some properties of this majorization. Then we try find some conditions on the structure of all linear operators which are preservers of this majorization.

Keywords: Linear preserver, Strictly sub row Hadamard majorization, Strictly sub row stochastic matrix.

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1 Introduction

Let $\mathbf{M}_{m,n}$ be the set of all m-by-n real matrices. A matrix C in $\mathbf{M}_{m,n}$ with nonnegative entries is called strictly sub column stochastic if the sum of entries on every column of C is less than 1. For $A, B \in \mathbf{M}_{m,n}$, we say that A is strictly sub column Hadamard majorized by B (denoted by $A \prec_{SCH} B$) if there exists an m-by-n strictly sub row stochastic matrix C such that $A = C \circ B$ where $X \circ Y$ is the Hadamard product (entrywise product) of matrices $X, Y \in \mathbf{M}_{m,n}$. In this paper, we introduce the concept of strictly sub column Hadamard majorization as a relation on $\mathbf{M}_{m,n}$. The Hadamard product has been penetrated in many branches of mathematical sciences and other sciences such as linear algebra theory, programming languages, statistics, etc. See [1–4]. For $X, Y \in \mathbf{M}_{m,n}$, the Hadamard product (entrywise product) of $X = [x_{ij}]$ and $Y = [y_{ij}]$, is denoted by $X \circ Y$ and is defined by $X \circ Y = [x_{ij}y_{ij}]$.

Definition 1.1. Let $X, Y \in \mathbf{M}_{m,n}$. We say that X is *SCH-Hadamard majorized* by Y (denoted by $X \prec_{SCH} Y$), if there exists a strictly sub column stochastic matrix $C \in \mathbf{M}_{m,n}$ such that $X = C \circ Y$.

For a linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$, it is said that T preserves (resp. strongly preserves) SCH-Hadamard majorization if $T(X) \prec_{SCH} T(Y)$ whenever $X \prec_{SCH} Y$ (resp. $T(X) \prec_{SCH} T(Y)$ if and only if $X \prec_{SCH} Y$). In this paper, we characterize all linear operators on $\mathbf{M}_{m,n}$ that preserve SCH-majorization. The following convention will be fixed throughout the paper. $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ is the standard basis of $\mathbf{M}_{m,n}$. When we use E_{ij}, the positive integers i and j are either fixed or are understood from the context. The m-by-n matrix $\mathbf{J}$ is the matrix of all ones, $\mathbf{C}_{m,n}$ is the set of all m-by-n column stochastic matrices, and $\mathbf{sC}_{m,n}$ is the set of all m-by-n sub column stochastic matrices.

In the next proposition we investigate a useful result from [5]. For every $m \in \mathbb{N}$, let $\mathbb{N}_m = \{1, \ldots, m\}$.

*Speaker. Email address: a.askari@vru.ac.ir

Proposition 1.2. *[5, Theorem 2.6] Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. The following conditions are equivalent:*

(1) *$T(E_{pq}) \circ T(E_{rs}) = 0$ for every $1 \le p, r \le m$ and $1 \le q, s \le n$ with $(p, q) \ne (r, s)$.*

(2) *There exist a function $f : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{N}_m \times \mathbb{N}_n$ and a matrix $A \in \boldsymbol{M}_{m,n}$ such that for every $X = [x_{i,j}] \in \boldsymbol{M}_{m,n}$,*

$$T(X) = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \vdots & \vdots \\ x_{f(m,1)} \cdots & x_{f(m,n)} \end{pmatrix} \circ A, \tag{1}$$

where $x_{f(i,j)}$ means x_{pq} if $f(i, j) = (p, q)$.

2 Main Results

In this section, first we state and prove some properties of preservers of SCH-Hadamard majorization on $\mathbf{M}_{m,n}$. Then we give some examples of linear preservers of SCH-Hadamard majorization. Finally, we find some useful conditions of all linear operators on $\mathbf{M}_{m,n}$ which preserve SCH-Hadamard majorization. The next remark is helpful in the following.

Remark 2.1. The next results hold:

(i) Let $A \in \mathbf{M}_{m,n}$. $A \prec_{SCH} A$ if and only if $A = 0$.

(ii) A linear operator $X \mapsto T(X)$ on $\mathbf{M}_{m,n}$, preserves $\prec_{SCH}$ if and only if $X \mapsto PT(X)Q$ preserves $\prec_{SCH}$, where $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ are arbitrary permutation matrices.

(iii) For $A \in \mathbf{M}_{m,n}$ with no zero entries, the linear operator $X \mapsto T(X)$ is a linear preserver of $\prec_{SCH}$ if and only if the linear operator $X \mapsto T(X) \circ A$ is a linear preserver of $\prec_{SCH}$.

Now we give a useful proposition about linear preservers of $\prec_{SCH}$ on $\mathbf{M}_{m,n}$.

Proposition 2.2. *If $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ is a linear preserver of $\prec_{SCH}$, then $T(E_{pq}) \circ T(E_{rs}) = 0$, for every $1 \le p, r \le m$ and $1 \le q, s \le n$ with $(p, q) \ne (r, s)$.*

Proof. Assume if possible that $T(E_{pq}) \circ T(E_{rs}) \ne 0$ for some $(p, q) \ne (r, s)$. So $[T(E_{pq})]_{ij} = \lambda \ne 0$ and $[T(E_{rs})]_{ij} = \mu \ne 0$ for some $1 \le i \le m$ and $1 \le j \le n$. Let $Y = \frac{1}{\lambda} E_{pq} - \frac{1}{\mu} E_{rs}$. Set $X = C \circ Y$, where $C = [c_{ij}]$ is a strictly sub column stochastic matrix such that c_{pq} and c_{rs} are $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Now, $X \prec_{SCH} Y$ but $T(X) \not\prec_{SCH} T(Y)$, which is a contradiction. So $T(E_{pq}) \circ T(E_{rs}) = 0$, for all $1 \le p, r \le m$ and $1 \le q, s \le n$ with $(p, q) \ne (r, s)$. □

Definition 2.3. Let $A \in \mathbf{M}_{m,n}$. We say that A is dominated by a $(0, 1)$-column stochastic matrix if there exists a $(0, 1)$-column stochastic matrix $C \in \mathbf{M}_{m,n}$ such that $A = A \circ C$. The set of all matrices which are dominated by $(0, 1)$-matrices is denoted by $\Delta_{m,n}$.

The next theorem gives important properties of linear preservers of SCH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 2.4. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T preserves SCH-Hadamard majorization, then the following conditions hold:*

(1) *For every $1 \le p \le m$ and $1 \le q \le n$, $T(E_{pq}) \in \Delta_{m,n}$.*

(2) *For every $1 \le p, r \le m$ and $1 \le q, s \le n$ with $p \ne r$, $T(E_{pq})$ and $T(E_{rs})$ do not simultaneously have a nonzero entry in any column.*

Proof. **(1)** Assume if possible that $T(E_{pq}) \notin \Delta_{m,n}$ for some $1 \leq p \leq m$ and $1 \leq q \leq n$. So by using part (ii) and part (iii) of Remark 2.1, at least two entries of the first column of $T(E_{pq})$ are 1. Set $X = E_{pq}$ and $Y = 2E_{pq}$. Thus, $X \prec_{SCH} Y$ but $T(X) \not\prec_{SCH} T(Y)$.

(2) Assume that $1 \leq p, r \leq m$, $1 \leq q, s \leq n$ with $p \neq r$ and let $T(E_{pq}) = [a_{ij}]$, $T(E_{rs}) = [b_{ij}]$. By part (ii) of Remark 2.1, without loss of generality we may assume that $a_{11} \neq 0$. Now by using Proposition 2.2, $b_{11} = 0$. We show $b_{i1} = 0$ for all $2 \leqslant i \leqslant m$. Let $b_{i1} \neq 0$ for some $2 \leq i \leq m$. Put $X = E_{pq} + E_{rs}$ and $Y = 2X$. So $X \prec_{SCH} Y$. We show that $T(X) \not\prec_{SCH} T(Y)$. If $T(X) \prec_{SCH} T(Y)$ there exists a strictly sub column stochastic matrix C such that

$$\begin{pmatrix} a_{11} & \dots & b_{1j} & \dots & \star \\ & \vdots & \vdots & \vdots & \\ \star & \dots & \star & \dots & \star \end{pmatrix}^t = C \circ \begin{pmatrix} 2a_{11} & \dots & 2b_{1j} & \dots & \star \\ & \vdots & \vdots & \vdots & \\ \star & \dots & \star & \dots & \star \end{pmatrix}^t,$$

which is imposible.

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By using Proposition 1.2, we can prove the following theorem.

Theorem 2.5. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T preserves SCH-Hadamard majorization, then the following conditions hold:*

(1) *There exist a function $f : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{N}_m \times \mathbb{N}_n$ and a matrix $A \in \boldsymbol{M}_{m,n}$ such that for every $X = [x_{i,j}] \in \boldsymbol{M}_{m,n}$,*

$$T(X) = \begin{pmatrix} x_{f(1,1)} & \dots & x_{f(1,n)} \\ \vdots & \vdots & \vdots \\ x_{f(m,1)} & \dots & x_{f(m,n)} \end{pmatrix} \circ A, \tag{2}$$

where $x_{f(i,j)}$ means x_{pq} if $f(i,j) = (p,q)$.

(2) *$T(X \circ Y) = T(X) \circ T(Y)$ for all $X, Y \in \boldsymbol{M}_{m,n}$ if $T(\boldsymbol{J})$ is a $(0,1)$-matrix.*

Proof. **(1)** Since T is a linear preserver of $\prec_{SCH}$, by using Proposition 2.2, we have $T(E_{pq}) \circ T(E_{rs}) = 0$ for every $1 \leq p, r \leq m$ and $1 \leq q, s \leq n$ with $(p,q) \neq (r,s)$. Now the conclusion follows from the Proposition 1.2.

(2) Assume that T is a linear preserver of $\prec_{SCH}$ and $T(\mathbf{J})$ is a $(0,1)$-matrix. By using Proposition 2.2, $T(E_{pq}) \circ T(E_{rs}) = 0$ for all $(p,q) \neq (r,s)$. So $T(E_{ij})$ is a $(0,1)$-matrix for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $T(E_{ij}) \circ T(E_{ij}) = T(E_{ij})$. Let $X = \sum_{i,j} x_{ij} E_{ij}$ and $Y = \sum_{i,j} y_{ij} E_{ij}$ be arbitrary m-by-n real matrices. Now we have

$$\begin{aligned} T(X \circ Y) =& T(\sum_{i,j} x_{ij} E_{ij} \circ \sum_{i,j} y_{ij} E_{ij}) \\ =& T(\sum_{i,j} x_{ij} y_{ij} E_{ij}) \\ =& \sum_{i,j} x_{ij} y_{ij} T(E_{ij}) \\ =& \sum_{i,j} x_{ij} T(E_{ij}) \circ \sum_{i,j} y_{ij} T(E_{ij}) \\ =& T(X) \circ T(Y). \end{aligned}$$

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To understanding the structure of the linear preservers of SCH-Hadamard majorization, we present the following examples.

Example 2.6. Assume that P is an m-by-m permutation matrix, Q is an n-by-n permutation matrix and $A \in \mathbf{M}_{m,n}$. The linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ defined by $T(X) = (PXQ) \circ A$ is a preserver of $\prec_{SCH}$.

Example 2.7. Let $X = [x_{ij}] \in \mathbf{M}_{m,n}$. Consider the linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ defined by

$$T(X) = \begin{pmatrix} x_{11} & x_{11} & 0 & \cdots & 0 \\ 0 & & & & \\ & & \ddots & & \\ & & & & \\ 0 & & & & 0 \end{pmatrix}^t .$$

Now, $I \prec_{SCH} 2I$ but $T(I) \not\prec_{SCH} T(2I)$. So T is not a preserver of $\prec_{SCH}$.

The following proposition is used to prove the main theorem of this section. For a subset X of $\mathbf{M}_{m,n}$, the set of extreme points of X is denoted by $\mathrm{ext}(X)$.

Proposition 2.8. *The set of all m-by-n real strictly sub column stochastic matrices is a strictly convex set that its extreme points are m-by-n, $(0,1)$-column stochastic matrices, i.e.*

$$\mathrm{ext}(\boldsymbol{sC}_{m,n}) = \{A \in \boldsymbol{C}_{m,n} : A \text{ is a } (0,1)\text{-column stochastic matrix}\}.$$

The following theorem is the key to characterize the linear preservers of SCH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 2.9. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. Then T preserves $\prec_{SCH}$ if and only if T satisfies the following conditions:*

(1) $T(E_{rs}) \circ T(E_{pq}) = 0$ *for every* $1 \le p, r \le m$ *and* $1 \le q, s \le n$ *with* $(r, s) \neq (p, q)$.

(2) *For every* $C \in \mathrm{ext}(\boldsymbol{sC}_{m,n})$ *there exists a* $(0,1)$*-matrix* $Z \in \boldsymbol{M}_{m,n}$ *such that* $Z \circ T(\boldsymbol{J}) = 0$ *and* $Z + T(C) \in \Delta_{m,n}$.

3 Conclusion

Throughout the paper, at first we introduce the concept of SCH-Hadamard majorization as a relation on $\mathbf{M}_{m,n}$ and study some properties of this relation. After that, we state some helpful conditions to characterize the linear operators that are preservers of SCH-Hadamard majorization.

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