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The structure of the preservers of SCH-Hadamard majorization

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Abstract

In this paper we determine the structures of all the linear operators $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ which (Strongly) preserve SCH-Hadamard majorization, where $\mathbf{M}_{m,n}$ is the set of all the m-by-n real matrices.

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1 Introduction

For $X, Y \in \mathbf{M}_{m,n}$, the Hadamard product (entrywise product) of $X = [x_{ij}]$ and $Y = [y_{ij}]$, is denoted by $X \circ Y$ and is defined by $X \circ Y = [x_{ij}y_{ij}]$. A matrix C in $\mathbf{M}_{m,n}$ with nonnegative entries is called strictly sub column stochastic if the sum of entries on every column of C is less than 1. For more details and applications of Hadamard product, the reader can see [1–4]. In this paper, with using the Hadamard product and strictly sub column stochastic matrices, we introduce a relation on $\mathbf{M}_{m,n}$ which is called strictly sub row Hadamard majorization or in brief SCH-majorization. Let $X, Y \in \mathbf{M}_{m,n}$. We say that X is *SCH-Hadamard majorized* by Y (denoted by $X \prec_{SCH} Y$), if there exists a strictly sub column stochastic matrix $C \in \mathbf{M}_{m,n}$ such that $X = C \circ Y$.

A linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is said to be a preserver (resp. strongly preserver) of SCH-Hadamard majorization if $T(X) \prec_{SCH} T(Y)$ whenever $X \prec_{SCH} Y$ (resp. $T(X) \prec_{SCH} T(Y)$ if and only if $X \prec_{SCH} Y$). In this paper, we determine the structure all linear operators on $\mathbf{M}_{m,n}$ that preserve (resp. strongly preserve) SCH-majorization. In the rest of this paper. $\{E_{11}, E_{12}, \dots, E_{mn}\}$ is the standard basis of $\mathbf{M}_{m,n}$. When we use E_{ij}, the positive integers i and j are either fixed or are understood from the context. The m-by-n matrix $\mathbf{J}$ is the matrix of all ones, $\mathbf{C}_{m,n}$ is the set of all m-by-n column stochastic matrices, and $\mathbf{sC}_{m,n}$ is the set of all m-by-n sub column stochastic matrices.

For $A \in \mathbf{M}_{m,n}$, we say that A is dominated by a $(0, 1)$-column stochastic matrix if there exists a $(0, 1)$-column stochastic matrix $C \in \mathbf{M}_{m,n}$ such that $A = A \circ C$. The set of all matrices which are dominated by $(0, 1)$-matrices is denoted by $\Delta_{m,n}$.

In the next proposition we investigate a useful result from [5]. For every $m \in \mathbb{N}$, let $\mathbb{N}_m = \{1, \dots, m\}$.

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Proposition 1.1. *[5, Theorem 2.6] Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. The following conditions are equivalent:*

(1) $T(E_{pq}) \circ T(E_{rs}) = 0$ *for every* $1 \leq p, r \leq m$ *and* $1 \leq q, s \leq n$ *with* $(p, q) \neq (r, s)$.

(2) *There exist a function* $f : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{N}_m \times \mathbb{N}_n$ *and a matrix* $A \in \boldsymbol{M}_{m,n}$ *such that for every* $X = [x_{i,j}] \in \boldsymbol{M}_{m,n}$,

$$T(X) = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \vdots & \vdots \\ x_{f(m,1)} \cdots & x_{f(m,n)} \end{pmatrix} \circ A, \tag{1}$$

where $x_{f(i,j)}$ *means* x_{pq} *if* $f(i, j) = (p, q)$.

2 Main Results

At the first of this section, we state some properties of preservers of SCH-Hadamard majorization on $\mathbf{M}_{m,n}$. Then we give an example of linear preservers and strong linear preservers of SCH-Hadamard majorization. Finally, we find the structure of all linear operators on $\mathbf{M}_{m,n}$ which preserve (strongly) preserve SCH-Hadamard majorization. The next remark is helpful in the following.

Remark 2.1. The next results hold:

(i) A linear operator $X \mapsto T(X)$ on $\mathbf{M}_{m,n}$, preserves $\prec_{SCH}$ if and only if $X \mapsto PT(X)Q$ preserves $\prec_{SCH}$, where $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$ are arbitrary permutation matrices.

(ii) For $A \in \mathbf{M}_{m,n}$ with no zero entries, the linear operator $X \mapsto T(X)$ is a linear preserver of $\prec_{SH}$ if and only if the linear operator $X \mapsto T(X) \circ A$ is a linear preserver of $\prec_{SH}$.

(iii) If $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is a linear preserver of $\prec_{SCH}$, then $T(E_{pq}) \circ T(E_{rs}) = 0$, for every $1 \leq p, r \leq m$ and $1 \leq q, s \leq n$ with $(p, q) \neq (r, s)$.

The next theorem gives important properties of linear preservers of SCH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 2.2. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T preserves SCH-Hadamard majorization, then the following conditions hold:*

(1) *For every* $1 \leq p \leq m$ *and* $1 \leq q \leq n$, $T(E_{pq}) \in \Delta_{m,n}$.

(2) *For every* $1 \leq p, r \leq m$ *and* $1 \leq q, s \leq n$ *with* $p \neq r$, $T(E_{pq})$ *and* $T(E_{rs})$ *do not simultaneously have a nonzero entry in any column.*

By using Proposition 1.1, we can prove the following theorem.

Theorem 2.3. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. If T preserves SCH-Hadamard majorization, then the following conditions hold:*

(1) *There exist a function* $f : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{N}_m \times \mathbb{N}_n$ *and a matrix* $A \in \boldsymbol{M}_{m,n}$ *such that for every* $X = [x_{i,j}] \in \boldsymbol{M}_{m,n}$,

$$T(X) = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \vdots & \vdots \\ x_{f(m,1)} \cdots & x_{f(m,n)} \end{pmatrix} \circ A, \tag{2}$$

where $x_{f(i,j)}$ *means* x_{pq} *if* $f(i, j) = (p, q)$.

(2) $T(X \circ Y) = T(X) \circ T(Y)$ *for all* $X, Y \in \boldsymbol{M}_{m,n}$ *if* $T(\boldsymbol{J})$ *is a* $(0,1)$*-matrix.*

The following example, is helpful to find the structure of the linear preservers of SCH-Hadamard majorization.

Example 2.4. Assume that P is an m-by-m permutation matrix, Q is an n-by-n permutation matrix and $A \in \mathbf{M}_{m,n}$. The linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ defined by $T(X) = (PXQ) \circ A$ is a preserver of $\prec_{SCH}$. Also, T strongly preserves $\prec_{SCH}$ if A has no zero entry. But $T(X) = (PX^tQ) \circ A$ is not a preserver of $\prec_{SCH}$ (X^t is the transpose of X).

The following theorem is the key to characterize the linear preservers of SCH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 2.5. *Let* $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ *be a linear operator. Then* T *preserves* $\prec_{SCH}$ *if and only if* T *satisfies the following conditions:*

(1) $T(E_{rs}) \circ T(E_{pq}) = 0$ *for every* $1 \leq p, r \leq m$ *and* $1 \leq q, s \leq n$ *with* $(r, s) \neq (p, q)$*.*

(2) *For every* $C \in \mathrm{ext}(\boldsymbol{sC}_{m,n})$ *there exists a* $(0,1)$*-matrix* $Z \in \boldsymbol{M}_{m,n}$ *such that* $Z \circ T(\boldsymbol{J}) = 0$ *and* $Z + T(C) \in \Delta_{m,n}$*.*

In the next Theorem we completely determine the structure of the linear operators $T : \mathbf{M}_{m,n} \to \mathbf{M}_{n,m}$, which preserves SCH-Hadamard majorization.

Theorem 2.6. *Let* $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ *be a linear operator. Then* T *preserves* $\prec_{SCH}$ *if and only if there exist* $A \in \boldsymbol{M}_{m,n}$ *and permutation matrices* $Q_1, \ldots, Q_n \in \boldsymbol{M}_m$ *such that*

$$T(X) = \begin{pmatrix} Y_{i_1}^t Q_1 \\ Y_{i_2}^t Q_2 \\ \vdots \\ Y_{i_n}^t Q_n \end{pmatrix}^t \circ T(\boldsymbol{J}), \qquad \forall X \in \boldsymbol{M}_{m,n}, \tag{3}$$

where Y_{i_j} *are some columnss of* X *for* $1 \leqslant j \leqslant n$ *(not necessarily distinct).*

2.1 Strong linear preservers of SCH-Hadamard majorization

In this section, we characterize the linear operators on $\mathbf{M}_{m,n}$ which strongly preserve SCH-Hadamard majorization. The next lemma shows that every strong linear preserver of $\prec_{SCH}$ on $\mathbf{M}_{m,n}$ is invertible.

Lemma 2.7. *Let* $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ *be a linear operator. If* T *strongly preserves* $\prec_{SCH}$*, then* T *is invertible.*

Proof. Assume that $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is a strong linear preserver of SCH-majorization and $T(X) = 0$. Then, $T(X) \prec_{SCH} 0$ and hence $X \prec_{SCH} 0$. Therefore, $X = 0$ which implies that T is invertible. □

Lemma 2.8. *Let* $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ *be a linear operator. If* T *strongly preserves* $\prec_{SCH}$*, then* $T(\boldsymbol{J})$ *has no zero entry.*

Proof. Assume that the linear operator $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ strongly preserves $\prec_{SCH}$. So by Theorem 2.3, T has the form 1.1 and by Lemma 2.7, T is invertible. Thus, $T(\mathbf{J})$ has no zero entry. □

The next proposision, gives necessary and sufficient conditions for a linear operator T on $\mathbf{M}_{m,n}$ that strongly preserves SCH-Hadamard majorization.

Proposition 2.9. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. Then T strongly preserves $\prec_{SCH}$ if and only if T is invertible and T satisfies the following conditions:*

(1) *$T(E_{rs}) \circ T(E_{pq}) = 0$ for every $1 \leq p, r \leq m$ and $1 \leq q, s \leq n$ with $(r, s) \neq (p, q)$.*

(2) *For every $C \in \mathrm{ext}(\boldsymbol{sC}_{m,n})$, $T(C)$ has exactly one nonzero entry in each column.*

The following theorem characterizes the linear preservers of SCH-Hadamard majorization on $\mathbf{M}_{m,n}$.

Theorem 2.10. *Let $T : \boldsymbol{M}_{m,n} \to \boldsymbol{M}_{m,n}$ be a linear operator. Then T strongly preserves $\prec_{SCH}$ if and only if there exist $A \in \boldsymbol{M}_{m,n}$ with no zero entry and permutation matrices $P \in \boldsymbol{M}_m$ and $Q_1, \ldots, Q_n \in \boldsymbol{M}_m$ such that*

$$T(X) = P \begin{pmatrix} Y_1^t Q_1 \\ Y_2^t Q_2 \\ \vdots \\ Y_n^t Q_n \end{pmatrix}^t \circ A, \qquad \forall X \in \boldsymbol{M}_{m,n}, \tag{4}$$

where $Y_1, \ldots, Y_n$ are columns of Y.

3 Conclusion

In this paper by state some useful results, we completely determine the structure of all the linear operators on $\mathbf{M}_{m,n}$ which preserve (strongly preserve) the SCH-Hadamard majorization.

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