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New inequalities on the weighted geometric mean of accreative matrices through non-standard domains

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Abstract

In this article we have proved some of the inequalities for accreative matrices through non-standard domains which have recently proved these inequalities for standard domains accreative matrices.

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1 Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathbb{M}_n$, we write that $A \geqslant 0$ if A is positive semidefinite, i.e. if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. We also write $A > 0$ if A is positive definite, i.e. if $\langle Ax, x\rangle > 0$ for all vectors $x \in \mathbb{C}^n$, and $A \geqslant B$ if $A - B \geqslant 0$.

A matrix $A \in \mathbb{M}_n$ is called accretive if in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}z + i\mathcal{I}z$, $\mathcal{R}z$ is positive definite, where $\mathcal{R}z = \frac{A+A^*}{2}$, $\mathcal{I}z = \frac{A-A^*}{2}$.
Later, Raissouli et. al. [4] defined the following weigheted geometric mean of two accretive matrices $A, B \in \mathbb{M}_n$,

$$A\sharp_\nu B = \frac{\sin \nu\pi}{\pi} \int_0^1 t^{\nu-1}(A^{-1} + tB^{-1})^{-1}\frac{dt}{t}$$

Raissouli et al. in [4] showed that if $A, B \in \mathbb{M}_n$ are accretive and $r \in (0, 1)$. Then

$$A\sharp_r B = B\sharp_{1-r} A \tag{1}$$

and

$$(\alpha A)\sharp_r(\beta B) = (\alpha\sharp_r\beta)(A\sharp_r B) \tag{2}$$

Bedrani et al. [2] showed if $A, B \in \mathbb{M}_n$ be accretive and $r \in (1, 2)$, then

$$A\sharp_r B = B(A\sharp_{2-r} B)^{-1}B \tag{3}$$

and if $r \in (-1, 0)$, then

$$A\sharp_r B = A(A\sharp_{-r} B)^{-1}A \tag{4}$$

For operator mean of accretive operators, we have the following result.

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Lemma 1.1. *[2] Let $A \in \mathbb{M}_n$ be accretive and $r \in (1,2)$. Then*

$$A\sharp_r B = \int_0^1 \left((1-s)B^{-1} + sB^{-1}AB^{-1}\right)^{-1} d\mu(s),$$
$$\text{where } \; d\mu(s) = \frac{\sin(r-1)\pi}{\pi}\frac{s^{r-2}}{(1-s)^{r-1}}ds.$$

Lemma 1.2. *[2] Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0)$. Then*

$$A\sharp_r B = \int_0^1 \left((1-s)A^{-1}BA^{-1} + sA^{-1}\right)^{-1} d\nu(s),$$
$$\text{where } \; d\nu(s) = \frac{\sin(r+1)\pi}{\pi}\frac{s^{r}}{(1-s)^{r+1}}ds.$$

Lemma 1.3. *[2] Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0) \cup (1,2)$. Then*

$$\mathcal{R}(A\sharp_r B) \leq \mathcal{R}A\sharp_r \mathcal{R}B$$

2 Main results

If $A \in \mathbb{M}_n$ are positive semidefinite and $r \in (0,1)$ is a real number, then equality $A\sharp_r B = B\sharp_{1-r}A$ is know. Raissouli et al. in [4] showed that if $A \in \mathbb{M}_n$ are accretive and $r \in (0,1)$, we still have $A\sharp_r B = B\sharp_{1-r}A$. In this section, I want to show that if $A \in \mathbb{M}_n$ are accretive and $r \in (-1,0) \cup (1,2)$, we also have this equality.

Proposition 2.1. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1,0) \cup (1,2)$. Then*

$$A\sharp_r B = B\sharp_{1-r}A$$

Proof. If $r \in (-1,0)$, then

$$\begin{aligned}
A\sharp_r B &= \int_0^1 \left((1-s)A^{-1}BA^{-1} + sA^{-1}\right)^{-1} d\nu(s) && \text{(by Lemma 1.2)}\\
&= \int_0^1 \left((1-s)A^{-1}BA^{-1} + sA^{-1}\right)^{-1} d\mu(s) &&\\
&= B\sharp_{1-r}A && \text{(by Lemma 1.1).}
\end{aligned}$$

□

Proposition 2.2. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (1,2)$. Then*

$$(\alpha A)\sharp_r B = \alpha^{r-1}(A\sharp_r B).$$

Proof. If $r \in (1,2)$, then $1-r \in (0,1)$ and we have

$$\begin{aligned}
\alpha A\sharp_r B &= B(\alpha A\sharp_{2-r}B)^{-1}B && \text{(by (3))}\\
&= B\left(\alpha^{r-1}(A\sharp_{2-r}B)\right)^{-1}B && \text{(by (2))}\\
&= \alpha^{r-1}B(A\sharp_{2-r}B)^{-1}B &&\\
&= \alpha^{r-1}(A\sharp_r B) && \text{(by (3)),}
\end{aligned}$$

□

Proposition 2.3. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1, 0)$. Then*

$$(\alpha A)\sharp_r B = \alpha^{r-1}(A\sharp_r B).$$

Proof. If $r \in (-1, 0)$, then $-r \in (0, 1)$ and we have

$$\begin{aligned}
\alpha A\sharp_r B &= \alpha A(\alpha A\sharp_{-r}B)^{-1}\alpha A && \text{(by (4))}\\
&= \alpha^2 A(\alpha A\sharp_{-r}B)^{-1}A\\
&= \alpha^2 A\left((\alpha^{1+r}A\sharp_{-r}B)\right)^{-1}A && \text{(by (2))}\\
&= \alpha^2\alpha^{-1-r}A(A\sharp_{-r}B)^{-1}A\\
&= \alpha^{1-r}A(A\sharp_{-r}B)^{-1}A\\
&= \alpha^{1-r}(A\sharp_r B) && \text{(by (4))}
\end{aligned}$$

□

Next, we present following theorem that is an analogue of [4, Proposition 4.1].

Theorem 2.4. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1, 0) \cup (1, 2)$. Then*

$$(\alpha A)\sharp_r(\beta B) = (\alpha\sharp_r\beta)A\sharp_r B.$$

Proof. Let $r \in (1, 2)$. Then

$$\begin{aligned}
(\alpha A)\sharp_r(\beta B) &= \alpha^{r-1}(A\sharp_r\beta B) && \text{(by Proposition 2.2)}\\
&= \alpha^{r-1}(\beta B\sharp_{1-r}A) && \text{(by Proposition 2.1)}\\
&= \alpha^{r-1}\beta^r(B\sharp_{1-r}A) && \text{(by Proposition 2.3)}\\
&= \alpha^{r-1}\beta^r(A\sharp_r B) && \text{(by Proposition 2.1).}
\end{aligned}$$

If $r \in (-1, 0)$, then $1 - r \in (1, 0)$ so proof is complete. □

Remark 2.5. Bakherad and Moslehian in [1] proved that if $a, b > 0$ and $r \notin [0, 1]$, then

$$ra + (1 - r)b \le a^r b^{1-r}, \tag{5}$$

so if $a, b > 0$ and $r \in (-1, 0) \cup (1, 2)$. Then

$$a!_r b = ((1 - r)a^{-1} + rb^{-1})^{-1} \ge (a^{-r}b^{r-1})^{-1} = a^{1-r}b^r. \tag{6}$$

Therefore if $A \in \mathbb{M}_n$ be accretive and $r \in (-1, 0) \cup (1, 2)$ by applied (5) and (6) we have

$$\mathcal{R}A\sharp_r\mathcal{R}B \le \mathcal{R}A!_r\mathcal{R}B,$$

finally by Lemma 1.3 we will have

$$\mathcal{R}(A\sharp_r B) \le \mathcal{R}A\sharp_r\mathcal{R}B \le \mathcal{R}A!_r\mathcal{R}B. \tag{7}$$

Remark 2.6. Fujii [3] proved that if $A, B \in \mathbb{M}_n^+$, then

$$\Phi(A\sharp_r B) \ge \Phi(A)\sharp_r\Phi(B), \quad r \in (-1, 0),$$

it holds, for any positive unital linear map Φ.
This inequality also holds for $r \in (1, 2)$, because by Proposition 2.1 if $r \in (1, 2)$ we have

$$\Phi(A\sharp_r B) = \Phi(B\sharp_{1-r}A) \ge \Phi(B)\sharp_{1-r}\Phi(A) = \Phi(A)\sharp_r\Phi(B).$$

Therefore if $A, B \in \mathbb{M}_n^+$ and $r \in (-1, 0) \cup (1, 2)$, then

$$\Phi(A \sharp_r B) \geq \Phi(A) \sharp_r \Phi(B),$$

in particular, with $C \geq 0$ and $\Phi(X) = Tr(XC)$, one has

$$Tr(C.(A \sharp_r B)) \geq Tr(CA) \sharp_r Tr(CB)$$

from which it follows that for every unitarily invariant norm $\|.\|$ we get

$$\|A \sharp_r B\| \geq \|A\| \sharp_r \|B\|. \tag{8}$$

As an application of Theorem 2.4, we have the following inequality.

Theorem 2.7. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1, 0) \cup (1, 2)$. Then*

$$\sum_{k=1}^{n} \langle (\Re(A \sharp_r B))^{-1} x_k, x_k \rangle \geq \left(\sum_{k=1}^{n} \langle (\Re A)^{-1} x_k, x_k \rangle \right) \sharp_r \left(\sum_{k=1}^{n} \langle (\Re B)^{-1} x_k, x_k \rangle \right),$$

for any family of vectors $(x_k)_{k=1}^n \in H$.

Proof. Using Theorem 2.4, (7) and a method similar to the proof of [4, Theorem 4.2] is proved. □

As a consequence of the Theorem 2.7, we have the following theorem.

Theorem 2.8. *Let $A \in \mathbb{M}_n$ be accretive and $r \in (-1, 0) \cup (1, 2)$. Then*

$$\| (\Re(A \sharp_r B))^{-1} \| \geq \|(\Re A)^{-1}\|^{1-r} \|(\Re B)^{-1}\|^r.$$

Proof. Taking the supremum over $\|x\| = 1$ of the latter inequality implies

$$\begin{aligned} \| (\Re(A \sharp_r B))^{-1} \| &= \sup \langle \Re(A \sharp_r B) x, x \rangle \\ &\geq \sup \langle (\Re A)^{-1} x_k, x_k \rangle \sharp_r \langle (\Re B)^{-1} x_k, x_k \rangle && \text{(by Theorem 2.7)} \\ &= \|(\Re A)^{-1} \sharp_r (\Re B)^{-1}\| \\ &\geq \|(\Re A)^{-1}\| \sharp_r \|(\Re B)^{-1}\| && \text{(by (8))} \\ &= \|(\Re A)^{-1}\|^{1-r} \|(\Re B)^{-1}\|^r. \end{aligned}$$

This completes the proof. □

3 Conclusion

In this paper, we have shown some of the inequalities that exist for accreative matrices that included the arithmetic mean of the standard domain in the opposite direction to the non-standard domain.

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