NUMERICAL RANGES OF EVEN-ORDER TENSOR

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Abstract. In this paper, the numerical range of an even-order tensor is defined using the norm of its square matrix unfolding. The basic properties of the numerical range of a matrix, such as compactness and convexity, are proved to hold for the numerical range of an even-order tensor. Also, we introduce normal tensors based on the contraction product. According to the Tucker decomposition, we get the numerical range of a normal tensor. Next, we introduce the singular-value decomposition (SVD) of an even-order tensor. Using this decomposition, we obtain the numerical range of such a tensor.
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1. INTRODUCTION

For a k^{th}-order l-dimensional tensor $\mathcal{A}$ (or $\mathcal{A} \in \mathbb{C}^{(k,l)}$), the numerical range is defined by

$$W_{|||.|||}(\mathcal{A}) = \bigcap_{\lambda \in \mathbb{C}} \{\mu \in \mathbb{C} : \mid \mu - \lambda \mid \leq \xi . ||| \mathcal{A} - \lambda \mathcal{I} |||\},$$

where $\mathcal{I} \in \mathbb{C}^{(k,l)}$ is the identity tensor, $|||\ .\ |||$ is a consistent tensor norm on $\mathbb{C}^{(k,l)}$, and ξ is a scalar [2]. One fundamental fact about the numerical range is that $W_{|||.|||}(\mathcal{A})$ is a convex subset of $\mathbb{C}$ that contains the spectrum of $\mathcal{A}$, for every $\mathcal{A} \in \mathbb{C}^{(k,l)}$. For high-dimensional problems, the data have an inherent tensor structure, and the difference slices of the data may have some relationships. If we process brain MRI images slice-by-slice, we may lose some information of the tensor structure of the images. Thus, it is necessary to study the tensor eigenvalue problem. The eigenvalues of high-order tensors are used in different fields. For example, medical resonance imaging, diffusion tensor imaging, high-order Markov chains and data mining, positive definiteness of even-order multivariate forms in automatic control, and best rank-one approximation in data analysis. Tensor eigenvalues were introduced by Lim and Qi in 2005.

Definition 1.1. *Let $\mathcal{A}$ be a k^{th}-order l-dimensional tensor. If $x \in \mathbb{C}^l \setminus \{0\}$ and $\lambda \in \mathbb{C}$ satisfy*

$$\mathcal{A}x^{k-1} = \lambda x^{[k-1]},$$

then we say that λ is an eigenvalue of $\mathcal{A}$, and that x is its corresponding eigenvector. Here,

$$(\mathcal{A}x^{k-1})_i := \sum_{i_2, i_3, \ldots, i_k = 1}^{l} a_{i, i_2, i_3, \ldots, i_k} x_{i_2} x_{i_3}, \ldots, x_{i_k},$$

where $1 \leq i \leq l$, $x = (x_1, x_2, \ldots, x_l)^T$, and $x^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_l^{k-1})$.

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It is clear that when $k = 2$, the above definition coincides with the one that defines the eigenvalues and eigenvectors of real matrices. Hence, tensor eigenvalues generalize matrix eigenvalues. According to Definition 1.1, a tensor eigenvalue problem is a nonlinear one which is equivalent to solving a set of multivariate polynomials of variables $x_1, x_2, \ldots, x_n$, and an unknown λ. In general, the tensor eigenvalue problem given by Definition 1.1 is very difficult.

Definition 1.2. *Let $\mathcal{A}$ be a $2m^{\text{th}}$-order n-dimensional tensor. Furthermore, assume that a nonzero m^{th}-order n-dimensional tensor $\mathcal{X}$ and $\lambda \in \mathbb{C}$ satisfy*

$$\mathcal{A} \cdot \mathcal{X} = \lambda \mathcal{X}, \tag{1}$$

where

$$(\mathcal{A} \cdot \mathcal{X})_{i_1,i_2,\ldots,i_m} = \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} x_{k_1,k_2\ldots,k_m},$$

and $1 \leq i_l \leq n, 1 \leq l \leq m$. Then, we refer to λ as an eigenvalue of $\mathcal{A}$, and to $\mathcal{X}$ as its corresponding eigentensor.

For $m = 1$, $\mathcal{A}$ is a square matrix and Definition 1.2 reduces to the definition of matrix eigenvalues and eigenvectors. In this paper, *eigenvalue* is always meant to be in the sense of Definition 1.2. The unfolding matrix of a tensor is a useful tool for the study of such tensor problems as those concerning eigenvalues and the numerical range.

Recently, *tensor numerical ranges* have been introduced by Ke, Li and Ng [2] on the basis of tensor norms. These have the same properties as those of the numerical ranges of matrices, except the *normality*, *projection*, and *unitary invariance* properties. The numerical ranges contain the eigenvalues. So, computing the numerical range of a tensor may be useful in designing fast algorithms for the calculation of its eigenvalues.

Our idea is to generalize the numerical range of a matrix to the numerical range of an even-order tensor, one that contains the tensor eigenvalues in the sense of Definition 1.2. We define the numerical range of even-order tensors using the even-order tensor unfolding matrix norms. We show that the basic properties of the numerical range of a matrix, such as *compactness* and *convexity*, are valid for the numerical range of an even-order tensor. It is useful to estimate Toeplitz tensor eigenvalues in the process of image restoration. Therefore, numerical ranges of Toeplitz tensors can be used in image processing.

2. Unfolding operations and tensors

Suppose that $\mathcal{A}$ is a $2m^{\text{th}}$-order n-dimensional tensor. We can reorder $\mathcal{A}$ as a square matrix using the square matrix unfolding of tensors.

Definition 2.1. *[1] Let $\mathcal{A}$ be a $2m^{\text{th}}$-order n-dimensional tensor. We use $\mathcal{A}(i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m)$ to denote the $(i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m)^{\text{th}}$ entry of $\mathcal{A}$. The square matrix unfolding of $\mathcal{A}$ with an ordering P is an n^m-by-n^m matrix A_P whose $(k, h)^{\text{th}}$ entry is given by*

$$A_P(k, h) = \mathcal{A}(i'_1, i'_2, \ldots, i'_m, j'_1, j'_2, \ldots, j'_m),$$

where

$$k = n^{m-1}(i'_1 - 1) + n^{m-2}(i'_2 - 1) + \cdots + n(i'_{m-1} - 1) + i'_m,$$
$$h = n^{m-1}(j'_1 - 1) + n^{m-2}(j'_2 - 1) + \cdots + n(j'_{m-1} - 1) + j'_m,$$

and $1 \leq i'_l, j'_l \leq n, 1 \leq l \leq m$. Here, P is the permutation matrix corresponding to ordering P that satisfies

$$(i'_1, i'_2, \ldots, i'_m) = (i_1, i_2, \ldots, i_m)P, \qquad (j'_1, j'_2, \ldots, j'_m) = (j_1, j_2, \ldots, j_m)P.$$

Example 2.2. *[1] Suppose that $\mathcal{A} = (a_{i,j,k,l})$ is a 4^{th}-order three-dimensional tensor. Let P be the permutation matrix given by*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that $(i'_1, i'_2) = (i_1, i_2)P$. The corresponding square matrix unfolding of $\mathcal{A}$ is a 3^2-by-3^2 matrix, namely,

$$A_P = \begin{bmatrix} a_{1111} & a_{1121} & a_{1131} & a_{1112} & a_{1122} & a_{1132} & a_{1113} & a_{1123} & a_{1133} \\ a_{2111} & a_{2121} & a_{2131} & a_{2112} & a_{2122} & a_{2132} & a_{2113} & a_{2123} & a_{2133} \\ a_{3111} & a_{3121} & a_{3131} & a_{3112} & a_{3122} & a_{3132} & a_{3113} & a_{3123} & a_{3133} \\ a_{1211} & a_{1221} & a_{1231} & a_{1212} & a_{1222} & a_{1232} & a_{1213} & a_{1223} & a_{1233} \\ a_{2211} & a_{2221} & a_{2231} & a_{2212} & a_{2222} & a_{2232} & a_{2213} & a_{2223} & a_{2233} \\ a_{3211} & a_{3221} & a_{3231} & a_{3212} & a_{3222} & a_{3232} & a_{3213} & a_{3223} & a_{3233} \\ a_{1311} & a_{1312} & a_{1331} & a_{1312} & a_{1322} & a_{1332} & a_{1313} & a_{1323} & a_{1333} \\ a_{2311} & a_{2321} & a_{2331} & a_{2312} & a_{2322} & a_{2332} & a_{2313} & a_{2323} & a_{2333} \\ a_{3311} & a_{3321} & a_{3331} & a_{3312} & a_{3322} & a_{3332} & a_{3313} & a_{3323} & a_{3333} \end{bmatrix}$$

Given two different orderings P and P', it is interesting to note that A_P and $A_{P'}$ are similar via a permutation matrix.

Proposition 2.3. *[1] Suppose that P and P' are two different orderings. Then, there exists a permutation matrix $\prod_{P,P'}$ such that*

$$\prod_{P,P'} A_P \prod_{P,P'}^T = A_{P'}.$$

Definition 2.4. *[1] Let $\mathcal{X}$ be an m^{th}-order n-dimensional tensor. The vectorization of $\mathcal{X}$ with an ordering P is an n^m-vector x_P whose j^{th} entry $x_P(j)$ is given by*

$$x_P(j) = \mathcal{X}_{i_1,i_2,\ldots,i_m}, \qquad 1 \le i_k \le n, 1 \le k \le m,$$

where $j = \sum_{k=1}^{m-1} n^{m-1}(i'_k - 1) + i'_m$ and P is the permutation matrix corresponding to the ordering P:$(i'_1, \ldots, i'_m) = (i_1, \ldots, i_m)P$.

Proposition 2.5. *[1] The tensor eigenvalue problem in (1) is equivalent to the matrix eigenvalue system*

$$A_P x_P = \lambda x_P.$$

According to Proposition 2.5, it is possible to calculate the eigenvalues and eigentensors of $\mathcal{A}$ by solving the eigenvalue problem of matrix unfolding A_P corresponding to $\mathcal{A}$ with ordering P.

Definition 2.6. *[1] Let $\mathcal{A}$ be a $2m^{\text{th}}$-order n-dimensional tensor. The tensor $\mathcal{A}^*$ whose entries are given by $\overline{a}_{j_1,j_2,\ldots,j_m,i_1,i_2,\ldots,i_m}$, for $1 \le i_k, j_k \le n$ and $1 \le k \le m$, is called the conjugate transpose of $\mathcal{A}$. We say that $\mathcal{A}$ is Hermitian if*

$$a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \overline{a}_{j_1,j_2,\ldots,j_m,i_1,i_2,\ldots,i_m}$$

for $1 \le i_k, j_k \le n$ and $1 \le k \le m$, that is, $\mathcal{A} = \mathcal{A}^*$. Note that $\mathcal{A}$ is symmetric if $a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = a_{j_1,j_2,\ldots,j_m,i_1,i_2,\ldots,i_m}$.

We define the *contraction product* of two $2m^{\text{th}}$-order n-dimensional tensors.

Definition 2.7. *[1] Let $\mathcal{A}$ and $\mathcal{B}$ be $2m^{\text{th}}$-order n-dimensional tensors. The contraction product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \star \mathcal{B}$, is a tensor of order $2m$ and dimension n which is defined by*

$$(\mathcal{A} \star \mathcal{B})_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} b_{k_1,k_2,\ldots,k_m,j_1,j_2,\ldots,j_m},$$

where $1 \le i_k, j_k \le n$ and $1 \le k \le m$.

Also, we can write

$$(\mathcal{A} \star \mathcal{B})(:, \ldots, :, j_1, \ldots, j_m) = \mathcal{A} \cdot \mathcal{B}(:, \ldots, :, j_1, \ldots, j_m),$$

where $1 \leq j_k \leq n, 1 \leq k \leq m$. The following proposition presents the basic properties of the contraction product.

Normal tensors. Let $\mathcal{A}$ be a $2m^{\text{th}}$-order n-dimensional tensor. We call $\mathcal{A}$ a *normal tensor* if $\mathcal{A} \star \mathcal{A}^* = \mathcal{A}^* \star \mathcal{A}$, that is,

$$\begin{aligned}(\mathcal{A} \star \mathcal{A}^*)_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} &= \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} \overline{a}_{j_1,j_2,\ldots,j_m,k_1,k_2,\ldots,k_m} \\ &= \sum_{k_1,k_2,\ldots,k_m=1}^{n} \overline{a}_{k_1,k_2,\ldots,k_m,i_1,i_2,\ldots,i_m} a_{k_1,k_2,\ldots,k_m,j_1,j_2,\ldots,j_m} \\ &= (\mathcal{A}^* \star \mathcal{A})_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m},\end{aligned}$$

where $1 \leq i_l, j_l \leq n$ and $1 \leq l \leq m$.

Diagonal tensor. A tensor $\mathcal{D} = (d_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m}) \in \mathbb{C}^{(2m,n)}$ is called a *diagonal tensor* if $d_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = 0$, when $(i_1, i_2, \ldots, i_m) \neq (j_1, j_2, \ldots, j_m)$.

Unitary tensors. A $2m^{\text{th}}$-order n-dimensional tensor $\mathcal{U}$ is said to be a *unitary tensor* if $\mathcal{U} \star \mathcal{U}^* = \mathcal{U}^* \star \mathcal{U} = \mathcal{I}_E$, where $\mathcal{I}_E = (e_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m})$ is the identity tensor in which

$$e_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \begin{cases} 1, & (i_1, i_2, \ldots, i_m) = (j_1, j_2, \ldots, j_m) \\ 0, & \textit{otherwise} \end{cases}$$

T-unitary similar tensors. Given $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{(2m,n)}$, we say that $\mathcal{A}$ is *T-unitary similar* to $\mathcal{B}$ if

$$\mathcal{A} = \mathcal{U}^* \star \mathcal{B} \star \mathcal{U},$$

where $\mathcal{U} \in \mathbb{C}^{(2m,n)}$ is a unitary tensor.

Theorem 2.8. *(Eigenvalue decomposition of normal tensors) Let $\mathcal{A}$ be a $2m^{\text{th}}$-order n-dimensional tensor. Then, $\mathcal{A}$ is a normal tensor if and only if there is a unitary tensor $\mathcal{V} \in \mathbb{C}^{(2m,n)}$ such that*

$$\mathcal{A} = \mathcal{V} \star \mathcal{D} \star \mathcal{V}^*,$$

where $\mathcal{D} \in \mathbb{C}^{(2m,n)}$ is a diagonal tensor whose entries are the eigenvalues of $\mathcal{A}$. Moreover, the above decomposition can be written as

$$\mathcal{A} = \sum_{i=1}^{n^m} \sigma_i \mathcal{V}_i \circ \mathcal{V}_i^*,$$

where $\mathcal{V}_i \in \mathbb{C}^{(m,n)}$ is the eigentensor corresponding to the eigenvalue σ_i.

Tensor norm. We define the *tensor norm* $|||\,.\,|||$ on $\mathbb{C}^{(m,n)}$ as follows. Given $\mathcal{X}$, an m^{th}-order n-dimensional tensor, we define its tensor norm by

$$|||\ \mathcal{X}\ ||| = \|x_P\|,$$

where $\|.\|$ is the vector norm.

Below, we provide some examples of tensor norms.

Let $\mathcal{X} = (a_{i_1,i_2,\ldots,i_m})$ be an m^{th}-order n-dimensional tensor.

- Tensor F-norm: $|||\ \mathcal{X}\ |||_F = \left(\sum_{i_1,i_2,\ldots,i_m=1}^{n} |\ a_{i_1,i_2,\ldots,i_m}\ |^2\right)^{\frac{1}{2}}$.
- Tensor 1-norm: $|||\ \mathcal{X}\ |||_1 = \|x_P\|_1$.

- Tensor ∞-norm: $|||\ \mathcal{X}\ |||_\infty = \|x_P\|_\infty$.
- Tensor $2-$norm: $|||\ \mathcal{X}\ |||_2 = \|x_P\|_2$.

Definition 2.9. *If $\|.\|$ is the matrix norm, the function $|||\ .\ |||_P$ defined on $\mathbb{C}^{(2m,n)}$ by*

$$|||\ \mathcal{A}\ |||_P = \|A_P\|$$

is called the tensor P-norm associated with the permuation matrix P. Also, $|||\ .\ |||_P$ is said to be consistent with the tensor norm on $\mathbb{C}^{(m,n)}$ if

$$|||\ \mathcal{A}\cdot\mathcal{X}\ ||| \leq |||\ \mathcal{A}\ |||_P\ |||\ \mathcal{X}\ |||,$$

where $\mathcal{X} \in \mathbb{C}^{(m,n)}$ and $|||\ .\ |||$ is the tensor norm.

The following examples are tensor P-norms consistent with the tensor norm $|||\ .\ |||$.

Let $\mathcal{A} = (a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m})$ be a $2m^{\text{th}}$-order n-dimensional tensor.

- Tensor P_F-norm: $|||\ \mathcal{A}\ |||_{P_F} = \left(\sum_{k_1,k_2,\ldots,k_m=1}^{n} \mid a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} \mid^2\right)^{\frac{1}{2}} = \|A_P\|_F$, where A_P is the square marix unfolding of $\mathcal{A}$. This tensor P_F-norm is consistent with the tensor norm $|||\ .\ |||_F$, because

$$|||\ \mathcal{A}.\mathcal{X}\ |||_F = \left(\sum_{i_1,i_2,\ldots,i_m=1}^{n} \mid \sum_{k_1,k_2,\ldots,k_m=1}^{n} a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} x_{k_1,k_2,\ldots,k_m} \mid^2\right)^{\frac{1}{2}}$$
$$\leq \left(\sum_{k_1,k_2,\ldots,k_m=1}^{n} \mid x_{k_1,k_2,\ldots,k_m} \mid^2\right)^{\frac{1}{2}}$$
$$\left(\sum_{i_1,i_2,\ldots,i_m=1}^{n} \sum_{k_1,k_2,\ldots,k_m=1}^{n} \mid a_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} \mid^2\right)^{\frac{1}{2}}$$
$$= |||\ \mathcal{X}\ |||_F |||\ \mathcal{A}\ |||_{P_F}\ .$$

- Tensor P_1-norm: $|||\ \mathcal{A}\ |||_{P_1} = \|A_P\|_1$.
- Tensor P_∞-norm: $|||\ \mathcal{A}\ |||_{P_\infty} = \|A_P\|_\infty$.
- Tensor P_2-norm: $|||\ \mathcal{A}\ |||_{P_2} = \|A_P\|_2$.

It is clear that $|||\ .\ |||_{P_1}, |||\ .\ |||_{P_2}$ and $|||\ .\ |||_{P_\infty}$ are consistent with the tensor norms $|||\ .\ |||_1, |||\ .\ |||_2$ and $|||\ .\ |||_\infty$.

Next, we let $\mathbb{F}$ be a subset of $\mathbb{C}^{(2m,n)}$ such that $\mathcal{A} = (a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m}) \in \mathbb{F}$ if and only if

$$a_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = a_{j_1,j_2,\ldots,j_m,k_1,k_2,\ldots,k_m}$$

for any $i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m$ and $(k_1, k_2, \ldots, k_m)$ being any permutation of $(i_1, \ldots, i_m)$. It is clear $\mathbb{F}$ is a vector space and the set of even-order symmetric tensors is a subset of $\mathbb{F}$.

Definition 2.10. *Let $|||\ .\ |||_{max}$ be a norm on $\mathbb{F}$ defined by*

$$|||\ \mathcal{A}\ |||_{max} = \max_{\mathcal{X}\neq 0} \frac{|||\ \mathcal{A}.\mathcal{X}\ |||}{|||\ \mathcal{X}\ |||},$$

where $|||\ .\ |||$ is a tensor norm on $\mathbb{C}^{(m,n)}$ and $|||\ \mathcal{A}.\mathcal{X}\ ||| = \|A_P x_P\|$.

This norm has the following properties.

- $||| \mathcal{I}_E |||=1$.
- For any $\mathcal{X} \in \mathbb{C}^{(m,n)}$,
$$||| \mathcal{A} \cdot \mathcal{X} ||| \leq ||| \mathcal{A} |||_{max} ||| \mathcal{X} |||.$$

The following examples are norms of this type.

- $||| \mathcal{A} |||_{max_1} = \max_{\mathcal{X} \neq 0} \frac{|||\mathcal{A}.\mathcal{X}|||_1}{|||\mathcal{X}|||_1}$.
- $||| \mathcal{A} |||_{max_2} = \max_{\mathcal{X} \neq 0} \frac{|||\mathcal{A}.\mathcal{X}|||_2}{|||\mathcal{X}|||_2}$.
- $||| \mathcal{A} |||_{max_\infty} = \max_{\mathcal{X} \neq 0} \frac{|||\mathcal{A}.\mathcal{X}|||_\infty}{|||\mathcal{X}|||_\infty}$.

Definition 2.11. *Let $\mathcal{A}$ be a $2m^{\text{th}}$-order n-dimensional tensor. The numerical range of $\mathcal{A}$ is defined by*
$$W_{|||\cdot|||_P}(\mathcal{A}) = \bigcap_{\lambda \in \mathbb{C}} \{\mu \in \mathbb{C} : | \mu - \lambda | \leq ||| \mathcal{A} - \lambda \mathcal{I}_E |||_P\},$$
where $||| \, . \, |||_P$ is a tensor P-norm consistent with the tensor norm, and $\mathcal{I}_E$ is a $2m^{\text{th}}$-order n-dimensional tensor.

Theorem 2.12. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be $2m^{\text{th}}$-order n-dimensional tensors. If $||| \, . \, |||_P$ is the tensor P-norm consistent with the tensor norm, then the following hold.*

(i) $W_{|||\cdot|||_P}$ $(\mathcal{A})$ *is a compact and convex set.*
(ii) $W_{|||\cdot|||_P}$ $(\mathcal{A} + \alpha \mathcal{I}_E) = W_{|||\cdot|||_P}$ $(\mathcal{A}) + \alpha$, *for any* $\alpha \in \mathbb{C}$.
(iii) $W_{|||\cdot|||_P}$ $(\alpha \mathcal{A}) = \alpha W_{|||\cdot|||_P}$ $(\mathcal{A})$, *for any* $\alpha \in \mathbb{C}$.
(iv) $W_{|||\cdot|||_P}$ $(\mathcal{A} + \mathcal{B}) \subset W_{|||\cdot|||_P}$ $(\mathcal{A})$ $+ W_{|||\cdot|||_P}$ $(\mathcal{B})$.
(v) *If* $\mathcal{A}, \mathcal{C} \in \mathbb{C}^{(m,n)}$, *then* $W_{|||\cdot|||_{max_2}}$ $(\mathcal{A} \oplus \mathcal{C}) = Co(W_{|||\cdot|||_{max_2}}$ $(\mathcal{A}) \cup W_{|||\cdot|||_{max_2}}$ $(\mathcal{C}))$.

Using the CP decomposition, we can find a relationship between numerical ranges and singular values of even-order tensors.

CP decomposition. [3] Let $\mathcal{A}$ be a $2m^{\text{th}}$-order n-dimensional tensor. If there exist a positive integer r, scalars α_j for $j \in [r]$, and vectors $x^{(j,i)}$ with $\|x^{(j,i)}\|_2 = 1$ for $i \in [2m]$ and $j \in [r]$ such that
$$\mathcal{A} = \sum_{j=1}^{r} \alpha_j x^{(j,1)} \otimes \cdots \otimes x^{(j,2m)},$$
then (**??**) is said be a CP decomposition of $\mathcal{A}$. It is easy to see $\mathcal{A}$ always admits such a tensor decomposition when r is sufficiently large . The minimal value of r is called the rank of $\mathcal{A}$.

Theorem 2.13. *If $\mathcal{A} \in \mathbb{C}^{(2m,n)}$ and $rank(\mathcal{A}) = m$, then*
$$\mathcal{A} = \mathcal{U} \star \mathcal{D} \star \mathcal{V}^*,$$
where $\mathcal{U}, \mathcal{V}, \mathcal{D} \in \mathbb{C}^{(2m,n)}$, and $\mathcal{D}$ is the diagonal tensor whose measure of the diagonal entries is equal to the square root of the eigenvalues of $\mathcal{A} \star \mathcal{A}^*$.

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