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An accelerated approach for low rank tensor completion

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Abstract

Tensor completion problem, which is a generalization of the matrix completion problem, is recovering the missing data of a tensor. In many algorithms proposed to complete the tensor, to achieve the answer, the method is executed on all unfolds related to tensor modes. Hence, if a tensor has N modes, each iteration of the algorithm contains N sub-problems, which is equivalent to solving N matrix completion problems. In this paper, to overcome the computational complexity caused by applying the algorithm to each dimension of the tensor, we present an idea in which the problem is implemented on only one tensor unfolding, so its computational complexity decreases in each iteration.

Keywords: Tensor completion problem, n-rank, computational complexity.

Mathematics Subject Classification [2010]: 15A83, 14N07

1 Introduction

Throughout the last decades, matrix representation of data and information has been arisen in many practical problems. In many fields of sciences such as machine learning, control, computer vision, collaborative filtering and dimensionality reduction, missing information is unavoidable in the data collection stage and a main important question is how we can fill these missing entries using known entries. This problem is known as matrix completion problem and since we can fill these unknown entries in different and infinite manners, this problem is ill-posed.
It is proved that by adding low-rankness property, this problem is no longer as an ill-posed problem and has an exact solution. In many cases, the structure of the data we want to recover is (approximately) low rank. So to solve this problem, we assume that the matrix is low rank. Following the data representation as the matrix, tensor, a powerful template for data management and matrix generalization to arbitrary order, was also considered. Data presentation in tensor form, exists in many applications such as neuroscience and chemistry.

Low-rank tensors play an important role in different real-world applications. A significant number of which, can be presented as data recovery problem (completion problem), including image/video inpainting, link-prediction and recommendation systems.

So far, different methods have been proposed to solve the low rank tensor completion problem. A class of these methods is based on unfolding the tensor in each tensor mode. Given the fact that N modes for the tensor will be equivalent to solving the N matrix completion problems,

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obviously, it requires a high memory, cost and computation time.
In this paper, to decrease the required memory and computational complexity, we propose an idea in which instead of solving the problem by using the tensor unfolding in all its modes, we solve the problem in only one unfolding. With this technique, in each iteration, we will need to solve the following problems related to only one tensor mode, and it is expected to create high reduction in the required time and memory.

2 Notations and definitions on tensors

Let $\mathcal{H}$ be the vector space of tensors in $\mathbb{R}^{I_1\times\cdots\times I_n}$. The inner product of $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{H}$ is defined as

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\cdots\sum_{i_N=1}^{I_N} x_{i_1,i_2,\ldots,i_N} y_{i_1,i_2,\ldots,i_N}, \tag{1}$$

and the Frobenius norm of tensor $\mathbf{X}$ is equal to

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\cdots\sum_{i_N=1}^{I_N} x^2_{i_1,i_2,\ldots,i_N}}. \tag{2}$$

A mode-n unfolding of tensor $\mathbf{X}$ is denoted by

$$X_{(n)} = \text{unfold}_n(\mathbf{X}) \in \mathbb{R}^{I_n\times\prod_{i\neq n} I_i}. \tag{3}$$

In this operator, the tensor element $(i_1, i_1, \ldots, i_N)$ is mapped to the matrix element (i_n, j), where

$$j = 1 + \sum_{\substack{k=1\\k\neq n}}^{N} (i_k - 1)J_k \quad \text{with} \quad J_k = \prod_{\substack{m=1\\m\neq n}}^{k-1} I_m.$$

The inverse operator of unfold is defined in such a way that

$$\mathbf{X} = \text{fold}_n X_{(n)}. \tag{4}$$

One of the most important and practical definitions for tensors is the definition of rank. Unlike the matrices, tensors don't have a unique rank definition. Different definitions have been proposed for the concept of tensor rank, one of which is known as n-rank based on Tucker decomposition and is defined for a N-mode tensor $\mathbf{X}$, as [5]

$$\text{n-rank}(\mathbf{X}) = \left(\text{rank}\left(X_{(1)}\right), \ldots, \text{rank}\left(X_{(N)}\right)\right). \tag{5}$$

In this article, we only focus on the n-rank and discuss low rank tensor completion problem based on n-rank definition.

3 Low rank tensor completion problem

The purpose of low rank tensor completion problem (LRTC) is to retrieve the unknown elements of the tensor according to the known entries, assuming the rank of the tensor is low. Under these conditions, the tensor completion problem is expressed as follows [1]

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} \ \text{n-rank}(\mathbf{X}) \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}), \end{aligned} \tag{6}$$

where $\mathbf{T}$ is the observation tensor, $\mathbf{X}$ is low rank approximation of $\mathbf{T}$, Ω is the set of all index pairs in $\mathbf{T}$ that $\mathbf{X}$ shares with $\mathbf{T}$ and $\mathcal{P}_\Omega(.)$ is the sampling operator with zeros at the positions not in Ω. By applying a function f to the problem 6, our interest minimization problem becomes

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} f(\text{n-rank}(\mathbf{X})) \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}), \end{aligned} \tag{7}$$

and by putting $f = \|.\|_1$, problem (7) is equivalent to

$$\begin{aligned} &\min \sum_{i=1}^{N} \text{rank}(X_{(i)}) \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{8}$$

Problem (8) is a non-convex, discrete and Np-hard problem and under suitable conditions, problem (8) is formally equivalent to its tightest convex relaxation based on the nuclear norm [1]. The following relaxed form of tensor completion problem is proposed as

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} \|\mathbf{X}\|_* \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{9}$$

The nuclear norm for tensors is defined as $\|\mathbf{X}\|_* = \sum_{i=1}^{N} \left\|X_{(i)}\right\|_*$, where $\|A\|_*$ is equal to singular values' sum of the matrix A. By applying this definition to problem (9) we have

$$\begin{aligned} &\min_{\mathbf{X}\in\mathcal{H}} \sum_{i=1}^{N} \left\|X_{(i)}\right\|_* \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{10}$$

There are several methods and algorithms for solving problem (10). In [6] a simple low rank tensor completion algorithm (**SiLRTC**) is proposed to solve (10). The high accuracy low rank tensor completion (**HiLRTC**) is another method proposed in this paper. In [2, 4] two similar methods are proposed based on low rank matrix factorization. In these methods, each mode unfolding is decomposed in two matrices. This idea (**TMac**) is improved in [4] by additional term to objective function in [2]. In [1] the generalization of schatten p-norm for tensor (**SpBCD**) is used as a better surrogate than nuclear norm when $p \to 0$.

By solving N sub-problems in each iteration of all mentioned algorithms, if the dimensions of the tensor are large, they are inefficient and impractical because of their computational cost. In this article, we present an idea to solve this problem, which we will discuss in the next section.

4 Proposed idea

In this article, we apply $\|.\|_\infty$ function on the rank function

$$f(\text{n-rank}(\mathbf{X})) = \|\text{n-rank}(\mathbf{X})\|_\infty = \max_{1\le j\le N} \text{rank}(X_{(j)}). \tag{11}$$

So the tensor completion problem (7) is considered as follows

$$\begin{aligned} &\min_{\mathbf{X}} \max_{1\le j\le N} \text{rank}(X_{(j)}) \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{12}$$

We show that the optimal value of problem (12) is an upper bound for the optimal value of problem (8).
Let $(\mathbf{X}^*, X^*_{(i)})$ where $\mathbf{X}^* \in \mathcal{H}$ and $X_{(i)} \in \mathbb{R}^{I_i \times \prod_{j\neq i}^N I_j}$ be the solution of problem (12). Then we have

$$X^*_{(i)} = \arg\min_{\mathbf{X}} \max_{1\leq j\leq N} \operatorname{rank}(X^*_{(j)}), \tag{13}$$

According to (13) we have

$$\operatorname{rank}(X^*_{(j)}) \leq \operatorname{rank}(X^*_{(i)}), \quad 1 \leq j \leq N,$$
$$\Longrightarrow \min \sum_{j=1}^{N} \operatorname{rank}(X^*_{(j)}) \leq N \times \operatorname{rank}(X^*_{(i)}).$$

Hence the solution of problem (12) is an upper bound of problem (8).

The main problem with (12) is that in practice the rank of the incomplete tensor is unknown to us, so it is impossible to determine in which mode, the matrix resulting from the unfolding of the tensor has the highest rank. To solve this problem, we make a small change in (12), and instead of minimizing the largest rank of tensor unfoldings, we minimize the rank of mode-i unfolding where $I_i = \max\{I_j, j = 1, \dots, N\}$. With this change, we introduce the new following problem

$$\begin{aligned} &\min_{X_{(i)}} \operatorname{rank}(X_{(i)}) \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}). \end{aligned} \tag{14}$$

As mentioned before, problem (14) is formally equivalent to its convex relaxation

$$\begin{aligned} &\min_{X_{(i)}} \ \|X_{(i)}\|_* \\ &\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{T}) \end{aligned} \tag{15}$$

where $I_i = \max\{I_j, j = 1, \dots, N\}$, such that they have accurately one unique solution [?]. Problem (15) can be solved by every proposed method applied to matrix completion problem and tensor completion method based on matrix unfolding.

5 Numerical results

In this section, we consider the experimental performance of solving problem (10) with four methods **HiLRTC** [6], **SiLRTC** [6], **SpBCD** [1] and **TMac** [2]. We display our accelerated idea implementation with these methods by **HiALRTC**, **SiALRTC**, **ASpBCD** and **ATMac**. We represent some experiments in setting the tensor completion and compare results of applying mentioned algorithms on our proposed problem (15) and problem (10) in terms of computation time and accuracy. The calculations were accomplished on a standard desktop computer by 32 GB of memory, an intel corei7 and 3 GHz processor.

For each test, we extract an index set Ω, uniformly at random, including the prescribed fraction of the total tensor data inputs and show this fraction by Sr such that

$$Sr = \frac{|\Omega|}{\prod_{i=1}^N I_j},$$

where $|\Omega|$ shows the cardinal number of the set Ω. The relative error estimation generated using each method is considered as the following function

$$\text{Error} = \frac{\|\mathcal{P}_{\Omega^c}(\mathbf{T} - \mathbf{X})\|_F}{\|\mathcal{P}_{\Omega^c}(\mathbf{T})\|_F}, \tag{16}$$

	$Sr = 0.3$		$Sr = 0.5$		$Sr = 0.8$	
Algorithm	Error	Time(s)	Error	Time(s)	Error	Time(s)
TMac	0.40	1.61	0.12	1.87	0.12	2.00
ATMac	0.14	0.51	0.24	0.64	0.31	0.82
Sp	0.08	2.88	0.06	2.89	0.04	2.84
ASp	0.11	1.05	0.08	0.40	0.07	0.13
Si	0.12	2.90	0.07	2.89	0.04	2.81
ASi	0.13	1.98	0.07	1.98	0.07	1.91
Hi	0.11	2.64	0.07	2.65	0.04	2.62
AHi	0.11	1.57	0.07	1.64	0.04	1.00

Table 1: The comparison results of applying the mentioned methods to (10) and our proposed problem (15), on a $187 \times 270 \times 3$ color image with 3 different values of Sr.

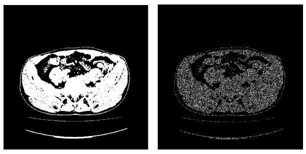

Figure 1: The front slice of an original medical image with 149 layers and its incomplete image with 50 percent of known entries as the input to the problem.

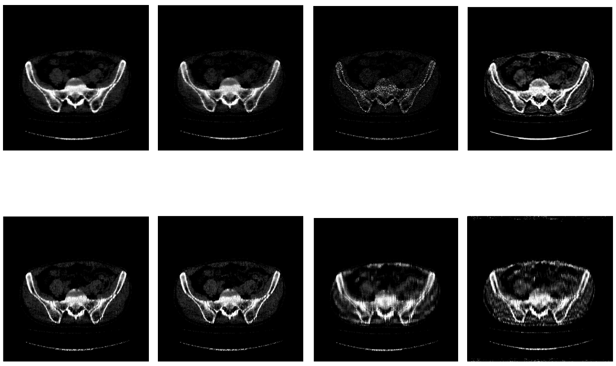

Figure 2: Recovery results of right picture in Figure 1. The first row is retrieved from left to right with algorithms **HiLRTC**, **SiLRTC**, **SpBCD** and **TMac** for solving problem (10) and the second row is recovered from left to right with **HiALRTC**, **SiALRTC**, **ASpBCD** and **ATMac**.

	$Sr = 0.3$		$Sr = 0.5$		$Sr = 0.8$	
Algorithm	Error	Time(s)	Error	Time(s)	Error	Time(s)
TMac	0.05	308.14	0.04	423.02	0.02	429.50
ATMac	0.12	108.84	0.10	104.68	0.07	104.87
Sp	0.07	238.57	0.06	251.88	0.02	242.93
ASp	0.12	16.04	0.10	17.35	0.11	16.14
Si	0.08	230.35	0.05	250.68	0.03	191.80
ASi	0.10	132.58	0.07	147.38	0.05	137.94
Hi	0.08	168.98	0.05	179.87	0.03	171.96
AHi	0.10	41.78	0.07	46.40	0.04	42.91

Table 2: The comparison results of applying the mentioned methods to (10) and our proposed problem (15), on a medical picture with 149 layers shown in Figure 1, with 3 different values of Sr.

where Ω^c is complementary to the corresponding sampling set Ω, $\mathbf{X}$ is the approximation of tensor completion and $\mathbf{T}$ is the original tensor. The error in each step of the algorithms is calculated as follows

$$\mathrm{E}_{tol} = \frac{\left\|\mathcal{P}_{\Omega^c}(\mathbf{X}^{k+1} - \mathbf{X}^k)\right\|_F}{\|\mathcal{P}_{\Omega^c}(\mathbf{X}^{k+1})\|_F}, \tag{17}$$

where $\mathbf{X^k}$ is the completion of incomplete tensor $\mathbf{W}$ at the kth iteration of the algorithm. The stopping criteria for each algorithm is considered as $\mathrm{E}_{rel} \leq \varepsilon$ where ε shows the desired error threshold and is set to 10^{-4} in our experiments.

For the subsequent experiments we focus on an color image data and an image of the radiological images set from the CPTAC Imaging Special Interest Group with 512×512 and 149 layers shown in Figure 1 and in order to apply the procedures, we removed some entries from the tensor data. Tables 1-2 and Figure 2 inform the comparison results of solving our represented problem (15) and problem (10) by the mentioned algorithms.

Considering the results obtained in Tables 1-2 and the comparison of the computation time of results obtained by mentioned algorithms on problem (10) and accelerated problem (15) show that the retrieval of images with the accelerated idea implementation by mentioned methods considerably reduces the computation time, while the resulted errors by solving problem (15) with mentioned methods are not significantly different from the errors obtained by applying the algorithms on problem (10) and in some cases, the errors are even less than those of results obtaind by solving problem (10). Generally, with a slight change in error and a significant reduction in the calculation time, especially for larger data, it can be said that the accelerated problem (15) significantly improves the complexity of solving problem (10).

6 Conclusion

In this paper, we presented a new idea with the approach of reducing the computational complexity and memory required to solve the tensor completion problem. Another advantage of the proposed idea is the ability to solve the tensor completion problem with various methods provided to solve the matrix completion problem.

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