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Construction of sensing matrices based on Rodin-Shapiro polynomials

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Abstract

For compressed sensing (CS) applications, it is significant to construct deterministic measurement matrices with good sparse recovery performance. In this article we present a method to obtain a class sensing matrices, by using Rudin-Shapiro polynomials.

Keywords: Rudin-Shapiro polynomials, Equiangular tight frames, compressive sensing, sparse recovery, Steiner system

Mathematics Subject Classification [2010]: 65F10, 15A29, 94A12, 42C40, 42A05

1 Introduction

Compressive sensing (CS), also known as compressive sampling, has received considerable research of interest in various applications due to its superior capability to recovery a sparse signal from a much smaller number of measurements than its original dimension. Mathematically speaking, given a measurement matrix (sensing matrix) $A \in R^{m\times N}$ with $m \ll N$, and given a measurement vector $\mathbf{y} = A\mathbf{x} \in R^m$ associated with an s-sparse vector $\mathbf{x} \in R^N$ (a vector that has at most s nonzero entries), we want to access this vector in a numerically tractable way. For solving CS problems, there are several classes of algorithms that have been used in applications, such as l_1 minimization algorithms, greedy algorithms, Iterative thresholding/shrinkage algorithms and combinatorial algorithms. In addition, a number of variants of the greedy pursuit algorithms have also been proposed by various authors, e.g., Orthogonal Matching Pursuit (OMP), Compressive Sampling Matching Pursuit (CoSaMP) and Subspace Pursuit (SP) (see [5]). Sensing matrix design is one important topic in compressive sensing. In this paper we consider the matrix sensing problem and present a class of measurements matrices, named Rudin-Shapiro equiangular tight frames (ETFs).

A family of vectors $F = \{f_i\}_{i=1}^N$ is a *frame* for a real M-dimensional Hilbert space $\mathbb{H}^M$ if there are constants $0 < A \le B < \infty$ so that for all $f \in \mathbb{H}^M$

$$A\|f\|^2 \le \sum_{i=1}^{N} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

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The corresponding *frame operator* is $FF^* = \sum_{n=1}^{N} f_n f_n^*$, where f_n^* denotes the linear functional that maps a given $f \in \mathbb{H}^M$ to the scalar $\langle f, f_n \rangle$.

The sequence F is said to be a *tight frame* if there exists $A > 0$ such that $FF^* = AI$. Meanwhile, F is *equiangular* if $\|f_n\| = 1$ for all n and if there exists $\alpha \geq 0$ such that $|\langle f_n, f_m \rangle| = \alpha$ for all $n \neq m$.

Many approaches to constructing ETFs have focused on the special case in which every entry of F is a root of unity [6]. In this article, we provide a construction of ETFs. A version of the ETF construction method we present here, was employed by M. Fickus and et. al. [4] (see also [1, 3] and references therein). To do this, at first we construct a class of discrete wavelet transforms, by applying the Rudin-Shapiro polynomials. These transformations as matrix representation, are orthogonal matrices. We apply these matrices with Steiner systems to present a class of ETFs, we named it Rudin-Shapiro ETFs..

This paper is organized as follows. In section 2 we introduce the classical Rudin-Shapiro polynomials to construct a family of discrete wavelet transforms, named Rudin-Shapiro DWT. In section 3, we construct tight frames with the introduced DWTs and finally the numerical simulation results are presented.

2 Classical Rudin-Shapiro polynomials

In this section we introduce a special type of trigonometric polynomials (as a ”pair”) called Rudin-Shapiro polynomials that will be used to construct a sequence of low-pass filters. They were introduced by H. S. Shapiro in his study of the magnitude for certain trigonometric sums. The Shapiro result was rediscovered by Rudin and published in 1959, and is now known as the Rudin-Shapiro polynomials. These polynomials have been used by many authors (see for example [2] and references therein).
If $P_0 \equiv 1$ and $Q_0 \equiv 1$, for $\xi \in [0, 1)$ and for all $n \geq 0$, we define the *Rudin-Shapiro polynomials* recursively by

$$P_{n+1}(\xi) = P_n(\xi) + e^{i2\pi 2^n \xi} Q_n(\xi), \tag{2.1}$$
$$Q_{n+1}(\xi) = P_n(\xi) - e^{i2\pi 2^n \xi} Q_n(\xi). \tag{2.2}$$

It can easily be verify that the coefficients of both P_n and Q_n are $+1$ or -1. If the sequence $\{\alpha_k\}_{k=0}^{\infty}$ in $\{-1, 1\}$ is so that

$$P_n(\xi) = \sum_{k=0}^{2^n-1} \alpha_k e^{2\pi i k \xi},$$

then $\alpha_0 = 1$ and for $k > 0$, we have $\alpha_{2k} = \alpha_k$ and $\alpha_{2k+1} = (-1)^k \alpha_k$. The following results can be found in [2].

Theorem 2.1. For any integer $n \geq 0$ and $0 \leq k \leq 2^{2n+1} - 1$, let

$$\alpha_n(k) = \frac{1}{2^{n+1}} \widehat{P}_{2n+1}(k) \text{ and } \beta_n(k) = (-1)^k \alpha_n(2^{2n+1} - 1 - k).$$

Then the functions φ_n and ψ_n which satisfy the following two-scale equations are father and mother wavelets, respectively,

$$\varphi_n(x) = \sqrt{2} \sum_{k=0}^{2^{2n+1}-1} \alpha_n(k) \varphi_n(2x - k), \tag{2.3}$$
$$\psi_n(x) = \sqrt{2} \sum_{k=0}^{2^{2n+1}-1} \beta_n(k) \phi_n(2x - k). \tag{2.4}$$

In the case of $n = 0, \varphi_0(x) = \varphi_0(2x) + \varphi_0(2x-1)$, which implies that $\varphi_0 = \chi_{[0,1)}$ and $\psi_0(x) = \varphi_0(2x) - \varphi_0(2x-1)$.
Now let $h(i)$ be a signal and $\alpha_n(k), \beta_n(k)$ be as in the previous theorem. Then the approximation operator H_n and the detail operator G_n corresponding to $\alpha_n(k)$ are defined by

$$(H_n h)(k) = \sum_i h(i)\overline{\alpha_n(i-2k)}, \quad (G_n h)(k) = \sum_i h(i)\overline{\beta_n(i-2k)}.$$

Therefore the matrix representation of these operators are as follows:

$$H_n = \begin{pmatrix} \alpha_n(0) & \alpha_n(1) & \alpha_n(2) & \dots & \alpha_n(M-2) & \alpha_n(M-1) \\ \alpha_n(M-2) & \alpha_n(M-1) & \alpha_n(0) & \dots & \alpha_n(M-4) & \alpha_n(M-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_n(2) & \alpha_n(3) & \alpha_n(4) & \dots & \alpha_n(0) & \alpha_n(1) \end{pmatrix}, \quad (2.5)$$

$$G_n = \begin{pmatrix} \beta_n(0) & \beta_n(1) & \beta_n(2) & \dots & \beta_n(M-2) & \beta_n(M-1) \\ \beta_n(M-2) & \beta_n(M-1) & \beta_n(0) & \dots & \beta_n(M-4) & \beta_n(M-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_n(2) & \beta_n(3) & \beta_n(4) & \dots & \beta_n(0) & \beta_n(1) \end{pmatrix} \quad (2.6)$$

Putting $M = 2^{2n+1}$, the Rudin-Shapiro discrete wavelet transform (DWT) is defined by

$$\mathbf{W}_n = \frac{1}{\sqrt{M}} \begin{pmatrix} H_n \\ G_n \end{pmatrix}$$

3 Constructing ETF

In [4], the authors have provided a new method for constructing equiangular tight frames (ETFs). The construction is based on a tensor-like combination of a Steiner system and a regular simplex.

Steiner system has been studied for over a century. In short, a Steiner system with parameters k, ν, written $(2, k, \nu)$-*Steiner system*, is an ν-element set V together with a set of order $\frac{\nu(\nu-1)}{k(k-1)}$ contains k-element subsets of V (called blocks) with the property that each 2-element subset of V is contained in exactly one block.

Here we employ DWT and Steiner system to construct tight frames.

Theorem 3.1. Every $(2, k, \nu)$-Steiner system generates a tight frame consisting of $N = \nu(1 + \frac{\nu-1}{k-1})$ vectors in $M = \frac{\nu(\nu-1)}{k(k-1)}$ dimensional space with redundancy $\frac{M}{N} = k(1 + \frac{k-1}{\nu-1})$ and density $\frac{k}{\nu} = (\frac{N-1}{M(N-M)})^{\frac{1}{2}}$.

Specifically, a $(\frac{\nu(\nu-1)}{k(k-1)}) \times \nu(1 + \frac{\nu-1}{k-1})$ tight frame matrix F may be constructed as follows:

(1) Let A^T be the $\frac{\nu(\nu-1)}{k(k-1)} \times \nu$ transpose of the adjacency matrix of a $(2, k, \nu)$-Steiner system;

(2) Let W be any $(1 + \frac{\nu-1}{k-1}) \times (1 + \frac{\nu-1}{k-1})$ matrix of corresponding to a DWT;

(3) For given $1 \le j \le \nu$, let F_j be the $\frac{\nu(\nu-1)}{k(k-1)} \times (1 + \frac{\nu-1}{k-1})$ matrix obtained from the j-th column of A^T by replacing each of the one-valued entries with a distinct rows of W, and every zero-valued entry with a row of zeros;

(4) $F = [F_1 F_2 \cdots F_\nu]$

In theorem (3.1) if W is a Hadamard matrix then the tight frame that we construct, will be equiangular [4]. For every positive integer n, $\mathbf{W}_n$ that constructed as a multiple of a Hadamard matrix. In fact, $2^{\frac{2n+1}{2}}\mathbf{W}_n$ is a Hadamard matrix, so we can put it instead of W in theorem(3.1) and get an equiangular tight frame.

For every $(2, k, \nu)$-Steiner system such that $\frac{\nu-1}{k-1} + 1 = 2^{2n+1}$ for some positive and integer n, let A^T be $\frac{\nu(\nu-1)}{k(k-1)} \times \nu$ transpose of the adjacency matrix of a $(2, k, \nu)$-Steiner system, and for each $j = 1, ..., \nu$ let F_j be a $\frac{\nu(\nu-1)}{k(k-1)} \times 2^{2n+1}$ matrix obtained from the j-th column of A^T by replacing each of the one-valued entries with a distinct row of the Hadamard matrix which is multiple of Golay wavelet transform. Then the columns of the $\frac{\nu(\nu-1)}{k(k-1)} \times \nu 2^{2n+1}$ matrix $F = [F_1...F_\nu]$, which have orthogonal rows and unit norm columns whose inner products have constant modulus α, provide a ETFs.

Example 3.1. The Ruding-Shapiro discrete wavelet transform corresponding to $n = 1$ is

$$\mathbf{W} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}$$

Let A^T be 28×8 transpose of the adjacency matrix of $(2, 2, 8)$-Steiner system, and for each $j = 1, ..., 8$ let F_j be 28×8 matrix obtained from the jth column of D^T by replacing each of the one-valued entries with a distinct row of $\sqrt{8}W$, and every zero-valued entry with a row of zeros. Then

$$F = \frac{4}{\sqrt{14}} \left[\begin{array}{ccc} F_1 & \dots & F_8 \end{array} \right]$$

is an equiangular tight frame consisting of 64 vectors in $\mathbb{R}^{28}$ with redundancy $\frac{16}{7}$ and density $\frac{1}{4}$.

Also suppose A^T is a 35×35 transpose of the adjacency matrix of $(2, 3, 15)$-Steiner system, and for each $j = 1, ..., 15$ let F_j be 35×8 matrix obtained from the jth column of A^T by replacing each of the one-valued entries with a distinct row of $\sqrt{8}W$, and every zero-valued entry with a row of zeros. Then

$$F = \frac{4}{\sqrt{14}} \left[\begin{array}{ccc} F_1 & \dots & F_{15} \end{array} \right]$$

is an equiangular tight frame consisting of 120 vectors in $\mathbb{R}^{35}$ with redundancy $\frac{24}{7}$ and density $\frac{1}{5}$.

4 Experimental Results

In Figure 1, we compare OMP and CoSaMP algorithms with both Rudin-Shapiro ETF and Gaussian measurement matrices $A \in R^{m\times n}$. The measurement signal is given by $y = Ax$. Reconstruction performance is quantified by the relative error, which is defined by

$$\text{relative error } = \frac{\|\tilde{x} - x\|_2}{\|x\|_2},$$

where $\tilde{x}$ is the reconstructed signal matrix and x is the original one.

The experiments illustrate how the relative error of each algorithm changes along the sparsity. We set $n = 120$ and $m = 35$. Let s (sparsity level) changes from 1 to 12 for measurement matrices. For each sparsity value ss, the algorithms are tested for 100 trials.

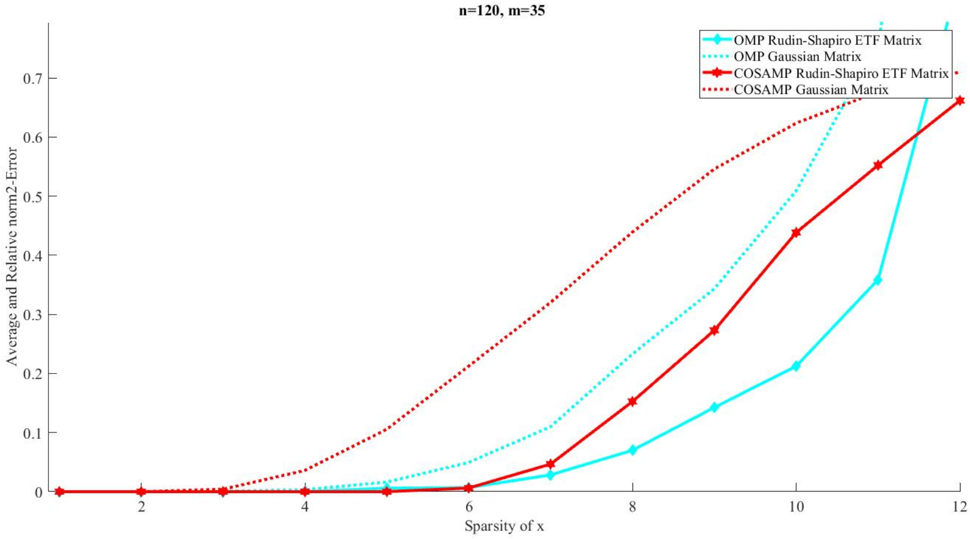

Figure 1: Plots of $\|x-x_0\|/\|x_0\|$ as a function for OMP and CoSaMP. These methods have the advantage at recovering Gaussian sparse vectors with Rudin-Shapiro ETF and Gaussian sensing matrices. The results are average of 100 runs.

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