The 11th International Seminar on
Linear Algebra and its Applications
Feb 2022

Convergence region of the generalized accelerated overrelaxation method for double saddle point problems

Mohammad Mahdi Izadkhah[1]*

[1]Department of Computer Science, Faculty of Computer and Industrial Engineering, Birjand University of Technology, Birjand, Iran

Abstract

This paper is devoted to generalize accelerated overrelaxation iterative method for solving a class of double saddle point problems. Also, we study convergence region of the proposed method and then some numerical results are given to demonstrate the efficiency of the presented method.

Keywords: AOR iterative method, Saddle point problem, convergence analysis

Mathematics Subject Classification [2010]: 65F08,65F10, 65F50

1 Introduction

We consider a class of double saddle point problems as the following large and sparse form

$$\mathcal{A}\mathbf{u} \equiv \begin{pmatrix} A & B & C \\ -B^T & 0 & 0 \\ -C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \equiv \mathbf{b}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite(SPD) matrix, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times p}$ have full column ranks, , $x, b_1 \in \mathbb{R}^n$, $y, b_2 \in \mathbb{R}^m$ and $z, b_3 \in \mathbb{R}^p$. For real eigenvalues of A, we use $\lambda_{min}(A)$ and $\lambda_{max}(A)$ to denote the minimum and maximum eigenvalues of A, respectively. Moreover, the notations $Ran(A)$ and $\rho(A)$ stand for the range and the spectral radius of A, respectively. Linear systems of the form (1) arise from mixed finite element approximation of potential fluid flow problems ; see [1, 2] and the references therein for detailed descriptions of these problems. The following Proposition gives necessary and sufficient conditions or the invertibility of the coefficient matrix $\mathcal{A}$ in (1).

Proposition 1.1. *Let A be SPD and assume that B and C have full column ranks. Then* $\mathrm{Ran}(B) \cap \mathrm{Ran}(C) = \{0\}$ *is a necessary and sufficient condition for the coefficient matrix $\mathcal{A}$ in* (1) *to be invertible.*

*Speaker. Email address: izadkhah@birjandut.ac.ir

2 Convergence region of DGAOR method

We propose generalized accelerated overrelaxation(AOR) iterative method for solving (1), based on the following splitting

$$\mathcal{A} = \mathcal{D} - \mathcal{E} - \mathcal{F},$$

where

$$\mathcal{D} = \begin{pmatrix} A & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & D \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ B^T & 0 & 0 \\ C^T & 0 & 0 \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 & -B & -C \\ 0 & Q & 0 \\ 0 & 0 & D \end{pmatrix}.$$

Here Q and D are preconditioning SPD matrices. The iteration matrix of the generalized AOR is defined as

$$\mathcal{M}_{r,\omega} = \begin{pmatrix} (1-\omega)I & -\omega A^{-1}B & -\omega A^{-1}B \\ \omega(1-r)Q^{-1}B^T & I - r\omega Q^{-1}B^TA^{-1}B & 0 \\ \omega(1-r)D^{-1}C^T & 0 & I - r\omega D^{-1}C^TA^{-1}C \end{pmatrix}. \tag{2}$$

Note that, if $\omega = 0$, then the proposed generalized AOR method diverges no matter what value the accelerated parameter r take. Thus, we will assume $\omega \neq 0$. Therefore, the generalized AOR method to solve double saddle point system (1) (DGAOR) can be defined by the following

$$\mathbf{u}^{(k+1)} = \mathcal{M}_{r,\omega}\mathbf{u}^{(k)} + \mathbf{c}, \quad k = 0, 1, 2, \ldots. \tag{3}$$

where

$$\mathbf{c} = \begin{pmatrix} A^{-1}b_1 \\ Q^{-1}(rB^TA^{-1}b_1 + b_2) \\ D^{-1}(rC^TA^{-1}b_1 + b_3) \end{pmatrix},$$

and $\mathbf{u}^{(0)} \in \mathbb{R}^{m+n+p}$ is the initial guess.

If we let λ be an eigenvalue of $\mathcal{M}_{r,\omega}$ and $\mathbf{u} = (x; y, z)^T$ be the corresponding eigenvector, then we have

$$(1 - \omega - \lambda)x = \omega A^{-1}(By + Cz), \tag{4}$$

$$(\omega - r + r\lambda)Q^{-1}B^Tx = (\lambda - 1)y, \tag{5}$$

$$(\omega - r + r\lambda)D^{-1}C^Tx = (\lambda - 1)z. \tag{6}$$

Lemma 2.1. *If we let λ be an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ of DGAOR method corresponding to the eigenvector $\mathbf{u} = (x; y, z)^T$, then x and z are not equal to zero, simultaneously, and $\lambda \neq 1$.*

Proof. If we set $x = 0$ and $z = 0$, then (4) implies that $By = 0$. Because B has full column rank, so $y = 0$ which is contradiction. Let $\lambda = 1$, and the associate eigenvector $\mathbf{u} = (x; y, z)^T$. Then, by equations (4)-(6) we have

$$A^{-1}(By + Cz) = -x, \quad Q^{-1}B^Tx = 0, \quad D^{-1}C^Tx = z.$$

This is the problem $\mathcal{A}\mathbf{u} = 0$, and by Proposition5 1.1 we have $\mathbf{u} = 0$, which is a contradiction. □

Lemma 2.2. *If $r = 1$, then $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of at least m. If $r \neq 1$, then $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ if and only if $n > m$; in this case the multiplicity of λ is $n - m - p$.*

Proof. By definition (2) of $\mathcal{M}_{r,\omega}$, it can be deduced that for $r = 1$, $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of at least m. Now, we assume $r \neq 1$. By equations (4)-(6) we have

$$(r-1)x^T BQ^{-1}B^T x = 0,$$

which implies $x = 0$ for $n = m$, and $x \neq 0$ for $n > m$. Thus $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of $n - (m + p)$. The latter statue comes from the fact that algebraic multiplicity of an eigenvalue is at least equal to geometrical multiplicity. □

Corollary 2.3. *Let λ be an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ and the associate eigenvector is $\mathbf{u} = (x; y, z)^T$. If $\lambda \neq 1 - \omega$, then $y \neq 0$ or $z \neq 0$.*

Theorem 2.4. *Let A be SPD and assume that B and C have full column ranks, such that* $\mathrm{Ran}(B) \cap \mathrm{Ran}(C) = \{0\}$*. If Q and D are preconditioning SPD matrices, then for parameters ω and r we have*

$$\lambda^2 + ((\omega - 2) + \frac{\gamma + \beta}{\alpha} r\omega)\lambda + (1 - \omega) + \frac{\gamma + \beta}{\alpha}\omega(\omega - r) = 0,$$

where $\alpha = x^* Ax$, $\beta = x^* BQ^{-1}B^T x$, and $\gamma = x^* CD^{-1}C^T x$.

Proof. Equations (4)-(6) give

$$\left(A + \frac{\omega(\omega - r + r\lambda)}{(1 - \omega - \lambda)(1 - \lambda)} CD^{-1}C^T\right) x = \frac{\omega}{1 - \omega - \lambda} By, \quad (\omega - r + r\lambda)Q^{-1}B^T x = (\lambda - 1)y,$$

and therefore by definitions of α, β and γ we have

$$(1 - \lambda)(1 - \omega - \lambda)\alpha + \omega(\omega - r + r\lambda)(\gamma + \beta) = 0$$

□

Lemma 2.5 ([4, Lemma 2.3]). *Both roots of the real quadratic equation $\lambda^2 - b\lambda + c = 0$, are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

It follows from Lemma 2.5 that $\rho(\mathcal{M}_{r,\omega}) < 1$ if and only if

$$|1 - \omega + \frac{\gamma + \beta}{\alpha}\omega(\omega - r)| < 1 \tag{7}$$

$$|\omega - 2 + \frac{\gamma + \beta}{\alpha} r\omega| < 1 + 1 - \omega + \frac{\gamma + \beta}{\alpha}\omega(\omega - r). \tag{8}$$

Equations (7) and (8) hold true if we have

$$0 < \omega < 2, \quad \omega - \frac{\alpha}{\gamma + \beta} < r < \frac{\omega}{2}.$$

Consequently, we have the following results.

Theorem 2.6. *Let A be SPD and assume that B and C have full column ranks, such that* $\mathrm{Ran}(B) \cap \mathrm{Ran}(C) = \{0\}$ *. If $0 < \omega < \min(2, \frac{2\lambda_{min}(A)}{\lambda_{max}(BQ^{-1}B^T) + \lambda_{max}(CD^{-1}C^T)})$ and*

$$\omega - \frac{\lambda_{min}(A)}{\lambda_{max}(BQ^{-1}B^T) + \lambda_{max}(CD^{-1}C^T)} < r < \frac{\omega}{2}$$

then the DGAOR iterative scheme (3) converges to the exact solution of (1).

3 Numerical results

We now describe some numerical experiments were carried out in order to analyze the behaviour of the DGAOR method for different values of the parameter ω and r. The computational study was done in the next problems.

Example 3.1. Let us consider the double saddle point system (1), where the matrices A, B and C are defined as follows

$$A = (a_{ij})_{n\times n} = \begin{cases} i+1, & i=j \\ 1, & |i-j|=1 \\ 0, & otherwise. \end{cases}, \quad B = (b_{ij})_{n\times m} = \begin{cases} j, & i=n-m+j \\ 0, & otherwise. \end{cases}$$

and

$$C = (c_{ij})_{n\times p} = \begin{cases} j, & i=j \\ 0, & otherwise. \end{cases}$$

For this problem we have that the conditions of Proposition 1.1 are satisfied(especially $\mathrm{Ran}(B)\cap \mathrm{Ran}(C) = \{0\}$), hence $\mathcal{A}$ is nonsingular and the double saddle point problem (1) has a unique solution. The vector $\mathbf{b}$ is chosen so that the components of the exact solution $\mathbf{u}$ of (1) have values equal to 1. We choose the preconditioning matrices $Q = B^TB$ and $D = C^TC$ for the DGAOR method. All runs are started from the initial zero vector and terminated if the current iterations satisfy $ERR = \frac{\|\mathbf{r}^{(k)}\|_2}{\|\mathbf{r}^{(0)}\|_2} \leq 10^{-4}$, or if the prescribed iteration number $k_{\max} = 2000$ is exceeded. Here we define $\mathbf{r}^{(k)}$ as

$$\mathbf{r}^{(k)} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} A & B & C \\ -B^T & 0 & 0 \\ -C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix}.$$

Figure 1 shows ERR in terms of ω and r for $n = 50, m = 30, p = 10$(left) and $n = 200, m = 100, p = 60$(right). In view of Figure 1 we can conclude the fact that the minimum ERR is obtained when the parameters ω and r are near the boundary of the convergence region. From the results recorded in Table 1 we can conclude that ERR and computational time are kind of important, to demonstrate the efficiency of DGAOR method in comparison with SOR-like method($\omega = r$) [3].

Table 1: CPU time, iteration number and values of the parameters for DGAOR method

					DGAOR method			SOR-like method		
n	m	p	ω	r	minIT	CPU(s)	ERR	minIT	CPU(s)	ERR
50	30	10	1.2538	0.5769	283	0.0313	9.902e-05	285	0.0625	9.996e-05
80	40	20	1.2538	0.5769	427	0.1719	9.980e-05	429	0.2188	9.978e-05
100	50	40	1.2538	0.5769	528	0.2813	9.918e-05	530	0.3281	9.914e-05
300	150	80	1.2538	0.5769	1571	4.2500	9.983e-05	1573	4.4688	9.983e-05

4 Conclusion

In this paper a generalization of accelerated overrelaxation (AOR) iterative method for solving a class of double saddle point problems has been proposed, denoted by DGAOR. The convergence region of the DGAOR method has been analyzed and numerical experiments were given to demonstrate the efficiency of DGAOR method in comparison with SOR-like method.

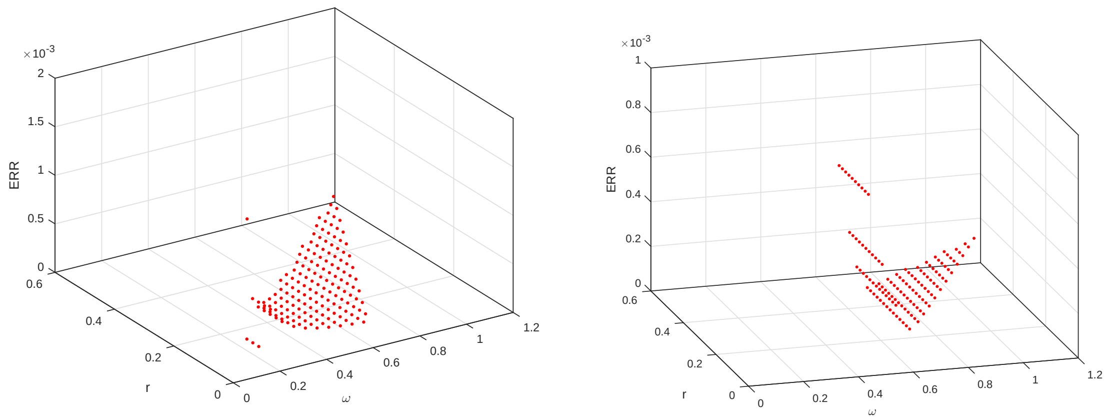

Figure 1: Convergence region for $n = 50, m = 30, p = 10$(left) and $n = 200, m = 100, p = 60$(right)

References

[1] F.P.A. Beik and M. Benzi, *Iterative methods for double saddle point systems*, SIAM J. Matrix Anal. Appl., **39**(2) (2018), 902–921.

[2] M. Benzi and F.P.A. Beik, Uzawa-type and augmented Lagrangian methods for double saddle point systems, *Structured Matrices in Numerical Linear Algebra*,Springer, Cham. (2019), 215–236.

[3] G.H. Golub, X. Wu, and J.-Y. Yuan, SOR-like methods for augmented systems, *BIT* 41 No. 1 (2001) 071–085.

[4] M.M. Martins, W. Yousif and J.L. Santos, A variant of the AOR method for augmented systems, *Mathematics of Computation* No. 277 (2012) 399–417.

[5] Y. Saad, *Iterative Methods for Sparse Linear Systems*, Second Edition, Society for Industrial and Applied Mathematics, Philadelphia, 2003.