SOLVABLE ULTRA-GROUPS

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Abstract. An ultra-group ${}_HM$ is a new structure algebra depends on a group G and its subgroup H. In the present paper we will investigate some identities characterizition of the solvable ultra-group and then investigate jordan-holder theorem in ultra-group.

1. Introduction

A pair of (A, B) of subsets of a group G is called transversal if the the equality $ab = a'b'$ implies $a = a'$ and $b = b'$, where $a, a' \in A$, $b, b' \in B$. this notation was introdused by Kurosh in [1] wich is the base of the concept of an ultragroup. This definition can be generalized for subgroups. It is not hard to deduce that a pair (H, M) of subgroups of G is transversal if and only if $H \cap M = \{e\}$. Moreover, for a subgroup H and a subset M of G we conclude that the pair (H, M) is a transversal if and only if $M \cap Hg$ contains at most one element, for all $g \in G$.
A subset M of G is called (right unitary) complementary set with respect to subgroup H, if for any elements $m \in M$ and $h \in H$ there exists unique elements $h^{'} \in H$ and $m^{'} \in M$ such that $mh = h^{'}m^{'}$ and $e \in M$. We denote h' and m' by ${}^mh$ and m^h, respectively.Similarly for any elements $m_1, m_2 \in M$ there exist unique elements $[m_1, m_2] \in M$ and $(m_1, m_2) \in H$ such that $m_1m_2 = (m_1, m_2)[m_1, m_2]$. For every element $a \in M$, there exists a^{-1} belongs to G. As $G = HM$, there is $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$. Now we are ready to define an ultra-group.
A (right) ultra-group ${}_HM$ is a complementary set of H over group G with a binary operation $\alpha :_H M \times_H M \rightarrow_H M$ and unary operation $\beta_h :_H M \rightarrow_H M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$.
A (left) ultra-group M_H is defined similarly via (left unitary) complementary set with respect to subgroup H and unary operation ${}_h\beta : M_H \rightarrow M_H$ defined by ${}_h\beta(m) := {}^hm$ for all $h \in H$. From now on, unless specified otherwise, ultra-group means right ultra-group.
Moreover, we exept one may consider the properties of the ultra-groups in justifying the associate theorems for the groups. All the notations in this paper is standard, we may refer the reader to see [2, 5, 6] for more details.

Definition 1.1. *Let ${}_HM$ be an ultra-group of subgroup H of a group G. A subset $S \subseteq {}_HM$ which contains the identity element of the group G, is called a subultra-group of ${}_HM$, if S is closed under operation α and β_h. This is denoted by $S <_H M$.*

Definition 1.2. *An ultra-group ${}_HM$ is called abelian, if for all elements a, b in ${}_HM$, $[a, b] = [b, a]$.*

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Proposition 1.3. $[[5], proposition 2.1]$ *Let $_HM$ be an ultra-group of subgroup H over the group G. Then we have the following properties:*

(i) $a^{hh'} = (a^h)^{h'}$,
(ii) $[a,b]^h = [a^{(^b h)}, b^h]$,
(iii) $[[a,b],c] = [a^{(b,c)}, [b,c]]$,
(iv) $e^h = e, \quad a^e = a$,
(v) $[e,a] = a = [a,e]$,
(vi) $[a^{[-1]}, a] = e = [a^{(a^{(-1)})}, a^{[-1]}]$,

for $a, b, c \in M$ and $h, h' \in H$.

Lemma 1.4. $[[5], lemma 2.1]$ *Let S be a subultra-group of ultra-group $_HM$ of H over the group G and $a, b \in {}_HM$. Then the following conditions are equivalent.*

(i) $a \in [S, b]$,
(ii) $[S, a] = [S, b]$,
(iii) $[a^{(b^{(-1)})}, b^{[-1]}] \in S$.

Definition 1.5. *Suppose $_{H_i}M_i$ is ultra-group of H_i over group G_i, $i = 1, 2$. A function $f :_{H_1} M_1 \longrightarrow_{H_2} M_2$ is an ultra-group homomorphism provided that for all $m, m_1, m_2 \in_{H_1} M_1$ and $h \in H_1$.*

(i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,
(ii) $(f(m))^{\varphi(h)} = f(m^h)$, *where φ is a group homomorphism between two subgroups H_1 and H_2.*

Definition 1.6. *A subultra-group N of $_HM$ is called normal if $[a, [N, b]] = [N, [a, b]]$, for all $a, b \in_H M$.*

According to the definition, every ultra-group such $_HM$, has subultra-group $\{e\}$ and ultra-group $_HM$ is normal subultra-group when the left cancellation low be established for $_HM$.

Theorem 1.7. *(First isomorphism theorem for ultra-groups) Let f be a surjective ultra-group homomorphism between two ultra-groups $_{H_1}M_1$ and $_{H_2}M_2$ and θ a congruence over $_{H_1}M_1$ such that $\theta \subseteq \mathrm{Ker}(f)$. If $\pi :_{H_1} M_1 \longrightarrow_{H_1} M_1/\theta$, then there exists a homomorphism g satisfying $g\pi = f$.*

Theorem 1.8. *(Second isomorphism theorem of ultra-groups)* $[[5], theorem 2.8]$ *If N', N are normal subultra-groups of ultra-group $_HM$ such that $N \subseteq N'$, then*

$$\frac{\frac{_HM}{N}}{\frac{N'}{N}} \cong \frac{_HM}{N'}.$$

2. SOLVABLE ULTRA-GROUPS

Similar to what was proved for the solvable groups, we obtain for ultra-groups and charactrization of solvable ultragroups as well as the famous theorem jordan-holder and Zassenhaus lemma.

Definition 2.1. *A subnormal series of an ultra-group $_HM$ is a chain of subultra-groups $_HM = M_0 > M_1 > \ldots > M_n$ such that M_{i+1} is normal subultra-group in M_i for $0 \le i \le n$. The factors of the series are the quotiont ultra-groups of stric inclusions(or alternatively, the number of nonidedtity factor).*

Definition 2.2. *A subnormal series of ultragroups such that M_i is normal in ${}_HM$ for all i is said to be normal series of ultra-groups.*

Every ultra-group ${}_HM$ has normal series such as ${}_HM > \{e\}$ since $\{e\}$ is normal subultra-group of ${}_HM$.

Lemma 2.3. *A subnormal series of ultra-groups need not be normal series of ultra-groups.*

Proof. Let $D_8 = \langle a, b | a^4 = b^2 = e, \quad (ab)^2 = e \rangle$ and $H = \{e\}$. The series $\{e\} < \{e, b\} < \{e, b, a^2, a^2b\} < D_8$ is subnormal series that it isn't normal series since $\{e, b\} \ntriangleleft D_8$. □

Definition 2.4. *Let ${}_HM = M_0 > M_1 > \ldots > M_n$ be subnormal series of ultra-groups. A one-step refinement of this series is any series of the form ${}_HM = M_0 > M_1 > \ldots > M_i > N > M_{i+1} >_n$ or ${}_HM = M_0 > M_1 > \ldots > M_i > N >_n > N$ where N is normal subulta-group of M_i and (if $i < n$) M_{i+1} is normal in N.*

Definition 2.5. *A refinement of subnormal series S is any subnormal series of ultra-groups obtaind from S by a finite sequence of one-step refinement.*

Definition 2.6. *An ultra-group ${}_HM$ is called simple if has just the normal subultra-group $\{e\}$.*

Definition 2.7. *A subnormal series ${}_HM = M_0 > M_1 > \ldots > M_n = e$ of ultra-group ${}_HM$ is composition series of ultra-groups if each factor $\dfrac{M_i}{M_{i+1}}$ is simple.*

Definition 2.8. *A subnormal series ${}_HM = M_0 > M_1 > \ldots > M_n = e$ of ultra-group ${}_HM$ is solvable series if each factor $\dfrac{M_i}{M_{i+1}}$ is abelian.*

Definition 2.9. *A maximal subultra-group S of an ultra-group ${}_HM$ is a proper subultra-group, such that no proper subultra-group K contains S strictly. Similary, a normal subultra-group N of ${}_HM$ is said to be a maximal proper normal subultra-group of ${}_HM$ if $N <_H M$ and there is no normal subultra-group K of ${}_HM$ such that $N < K <_H M$.*

If N is a normal subultra-group of an ultra-group ${}_HM$, then every subultra-group of $\dfrac{{}_HM}{N}$ is the form $\dfrac{K}{N}$ where K is subultra-group of ${}_HM$ containing N. In the other hand $\dfrac{K}{N}$ is normal subultra group of $\dfrac{{}_HM}{N}$ if and only if K is normal subultra-group of ${}_HM$. Thus, when ${}_HM \neq N$, $\dfrac{{}_HM}{N}$ is simple if and only if N is maximal in the set of all normal subultra groups L of ${}_HM$ with $L \neq_H M$. Such an ultra-group N is called a maximal subultra-group of ${}_HM$.

Theorem 2.10. *Let ${}_HM$ be an ultra-group of subgroup H over group G then we have the following properties:*
i) Every finite ultra-group ${}_HM$ has composition series.
ii) Every rifinement of a solvable series of ultra-group is a solvable series .
iii) A subnormal series of ultra-groups is a composition series if and only if it has no proper refinements.

Definition 2.11. *Two subnormal series S and T of an ultra-group ${}_HM$ are equvalent if there is a one to one correspondence between the nontrivial factors of S and the nontrivial factors of T such that corresponding factors are isomorphic ultra-groups.*

Two subnormal series need not have the same number of terms in order to be equivalent, but they have the same length (that is, the same number of nontrivial factors). Clearly, equivalence of subnormal series is an equivalence relation.

Lemma 2.12. *If S is a composition series of ${}_HM$, then any refinement of S is equvalent to S.*

In this section, we show that, similar Poropositions Zassenhaus and Schreier, Jordan-holder in group theory, it also holds in ultra-groups. Befor proving the Zassenhaus lemma we first prove some required lemma.

Lemma 2.13. *For every $a, b, c \in_H M$ and every subultra-group K of ${}_HM$ if $[a, b] = c$ and $a, c \in K$ then $b \in K$.*

Theorem 2.14. *If K, N is subultra-groups of ultra-group ${}_HM$ such that $N \lhd_H M$ then $N \cap K \lhd K$*

The next lemma is quite technical. It's value will be immediately apparent in the proof of theorem 2.16.

Lemma 2.15. ***(Zassenhaus)***
Let K^*, N^*, N, K be subultra-groups of ultra-group ${}_HM$ such that K^* is normal in K and N^* is normal in N. Then we have the following properties:
i) $[N^*, (N \cap K^*)] \lhd [N^*, (N \cap K)]$
ii) $[K^*, (N^* \cap K)] \lhd [K^*, (N \cap K)]$
iii) $\frac{[N^*,((N\cap K)]}{[N^*,(N\cap K^*)]} \cong \frac{[K^*,(N\cap K)]}{[K^*,(N^*\cap K)]}$

Theorem 2.16. ***(Schreier)***
Any two subnormal (resp. normal) series of an ultra-group ${}_HM$ have subnormal(resp. normal) refinements are equivalent.

Theorem 2.17. ***(Jorden-Holder)***
Any two composition series of an ultra-group ${}_HM$ are equivalent. There for every ultra-group having a composition series determines a unique list of simple ultra-groups.

Proof. Since composition series are subnormal series, any two composition series have equivalent refinement by theorem Schrier. But every refinement of a composition series S is equivalent to S by lemma 2.12. It follows that every two composition series are equivalent. □

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