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g-Fusion Frames in Hilbert Spaces

Sedigheh Hosseini*

Department of Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran

Abstract

In this paper, we investigate the notion of g-fusion frames in Hilbert spaces. Then, we present sufficient conditions for g-fusion frames in terms of g-frames in Hilbert spaces. We extend some of the recent results of standard g-frames and fusion frames to g-fusion frames.

Keywords: Hilbert space, Frame, Generalized Frames, G-fusion Frame.

Mathematics Subject Classification [2010]: $42C15$, $46B20$

1 Introduction

Frames for Hilbert space were first introduced by Duffin and Schaeffer [3] in 1952. Daubechies, Grossmann and Meyer [2] reintroduced frames, in 1986 [2] and considered from then. Frame theory has applications in signal processing, image processing, data compression and sampling theory.

Orthonormal bases are special case of frames in Hilbert space. Any element in Hilbert can be present as an infinite linear combination, not necessary unique, of the frame element.

Some new type and generalization of frame were introduced by researcher such as fusion frames, g-frames, woven frames, etc. Frame of subspaces or fusion frames are a generalization of frames which were introduced by Cassaza and Kutyniok [1] in 2003. Generalized frames or in abbreviation g-frames were introduced by Sun [6] in 2006. Most recently, g-fusion frames in Hilbert space were introduced by Sadri et.al. [5].

In this paper, motivated and inspired by the above-mentioned works we introduce the concept of g-fusion frame. This frame includes g-frames and fusion frames. We extend some of the recent results of standard g-frames and fusion frames to g-fusion frames.

The paper is organized as follows: Section 2 contains the basic definitions about fusion frames, g-frames, g-fusion frames. Section 3 belongs to the g-fusion frames and investigating their structuers.

2 Main results

As a preliminary of frames, at the first, we mention fusion frames. Also we review g-frames, g-fusion frames and woven frames . Through of this paper, $\mathcal{I}$ is the indexing set where it can be finite or infinity countable set. Also, $\mathcal{H}$ and $\mathcal{H}_i$ are separable Hilbert spaces and $B(\mathcal{H}, \mathcal{H}_i)$ is the collection of all the bounded linear operators of $\mathcal{H}$ into $\mathcal{H}_i$. If $\mathcal{H} = \mathcal{H}_i$, then $B(\mathcal{H}, \mathcal{H})$ will be denoted by $B(\mathcal{H})$ and P is the orthogonal projection.

*Speaker. Email address: rsana7238@gmail.com

2.1 Fusion frames

In 2003, a new type of generalization of frames was introduced by Cassaza and Kutyniok to the science world that today we know as fusion frames. In this section, we briefly recall some basic notations, definitions and some important properties of fusion frames that are useful for our study.

Definition 2.1. Let $\{v_i\}_{i\in\mathcal{I}}$ be a family of real weights such that $v_i > 0$ for all $i \in \mathcal{I}$. A family of closed subspaces $\{W_i\}_{i\in\mathcal{I}}$ of a Hilbert space $\mathcal{H}$ is called a fusion frame (or frame of subspaces) for $\mathcal{H}$ with respect to weights $\{v_i\}_{i\in\mathcal{I}}$, if there exist constants $C, D > 0$ such that

$$C||f||^2 \leq \sum_{i\in\mathcal{I}} v_i^2||P_{W_i}(f)||^2 \leq D||f||^2, \qquad \forall f \in \mathcal{H}, \tag{1}$$

where P_{W_i} is the orthogonal projection of $\mathcal{H}$ to W_i. The constants C and D are called the lower and upper fusion frame bounds, respectively. If the right inequality in (1) holds, the family of subspace $\{W_i\}_{i\in\mathcal{I}}$ is called a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in\mathcal{I}}$ with Bessel bound D. Also is called tight fusion frame with respect to $\{v_i\}_{i\in\mathcal{I}}$, if $C = D$ and is called Parseval fusion frame, if $C = D = 1$. We say $\{W_i\}_{i\in\mathcal{I}}$ an orthogonal fusion basis for $\mathcal{H}$, if $\mathcal{H} = \bigoplus_{i\in\mathcal{I}} W_i$.

Definition 2.2. The fusion frame $\{W_i\}_{i\in\mathcal{I}}$ with respect to some family of weights is called a Riesz decomposition of $\mathcal{H}$, if for every $f \in \mathcal{H}$, there is a unique choice of $f_i \in W_i$ so that $f = \sum_{i\in\mathcal{I}} f_i$.

For each family of subspaces $\{W_i\}_{i\in\mathcal{I}}$ of $\mathcal{H}$, the representation space:

$$\left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in\mathcal{I}} | f_i \in W_i \ \ and \ \ \sum_{i\in\mathcal{I}} ||f_i||^2 < \infty\right\},$$

with inner product

$$\langle\{f_i\}_{i\in\mathcal{I}}, \{g_i\}_{i\in\mathcal{I}}\rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i\rangle,$$

is a Hilbert space. This space is needed in the studying of fusion systems.

Definition 2.3. Let $\{W_i\}_{i\in\mathcal{I}}$ be a fusion frame family for $\mathcal{H}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$. Then the analysis operator for $\{W_i\}_{i\in\mathcal{I}}$ with weights $\{v_i\}_{i\in\mathcal{I}}$ is defined by:

$$U_{W,v} : \mathcal{H} \to \left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2}, \qquad U_{W,v}(f) = \{v_i P_{W_i}(f)\}_{i\in\mathcal{I}}.$$

The adjoint of $U_{W,v}$ is called the synthesis operator, we denote $T_{W,v} = U^*_{W,v}$.

By elementary calculation, we have

$$T_{W,v} : \left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_{W,v}(\{f_i\}_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} v_i P_{W_i} f_i.$$

Like discrete frames, the fusion frame operator for $\{W_i\}_{i\in\mathcal{I}}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$ is the composition of analysis and synthesis operators,

$$S_{W,v} : \mathcal{H} \to \mathcal{H}, \qquad S_{W,v}(f) = T_{W,v}U_{W,v}(f) = \sum_{i\in I} v_i^2 P_{W_i}(f), \quad \forall f \in \mathcal{H}.$$

The following theorem presents the equivalence conditions between the fusion frames and their operators.

Theorem 2.4. *Let $\{W_i\}_{i\in\mathcal{I}}$ be a family of subspaces in $\mathcal{H}$ and $\{v_i\}_{i\in\mathcal{I}}$ be a family of weights. Then the following conditions are equivalent:*

(i) The family $\{W_i\}_{i\in\mathcal{I}}$ is a fusion frame with respect to $\{v_i\}_{i\in\mathcal{I}}$,

(ii) The synthesis operator $T_{W,v}$ is bounded, linear and onto,

(iii) The analysis operator $U_{W,v}$ is a (possibly into) isomorphism.

2.2 Generalized frames

Sun [6] introduced g-frames which are generalized frames and include ordinary frames and many recent generalizations of frames.

Definition 2.5. Let $\{\mathcal{H}_i\}_{i\in\mathcal{I}}$ be a family of Hilbert spaces. We call $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i),\ i \in \mathcal{I}\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathcal{I}}$, or simply, a g-frame for H, if there exist two positive constants C, D such that

$$C||f||^2 \leq \sum_{i\in\mathcal{I}} ||\Lambda_i f||^2 \leq D||f||^2, \qquad \forall f \in \mathcal{H}. \tag{2}$$

The positive numbers C and D are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame, if $C = D$ and we call it a Parseval g-frame, if $C = D = 1$. If only the second inequality holds, we call it g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in\mathcal{I}} \Lambda_i^* \Lambda_i f, \qquad f \in \mathcal{H},$$

which is a bounded, positive and invertible operator such that

$$CI \leq S_\Lambda \leq DI,$$

and for each $f \in \mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in\mathcal{I}} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f = \sum_{i\in\mathcal{I}} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f.$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathcal{I}}$ with bounds $\frac{1}{D}, \frac{1}{C}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathcal{I}}$ and $\{\Lambda_i\}_{i\in\mathcal{I}}$ are dual g-frames with respect to each other.

It is easy to show that by letting $\mathcal{H}_i = W_i$, $\Lambda_i = P_{W_i}$ and $v_i = 1$, a fusion frame is a g-frame.

3 Generalized Fusion frames

Generalized fusion frames (g-fusion frames) in Hilbert space were introduced by Sadri et.al. [5].

Let

$$\left(\sum_{i\in\mathcal{I}} \oplus \mathcal{H}_i\right)_{\ell^2} = \left\{ \{f_i\}_{i\in\mathcal{I}} | f_i \in \mathcal{H}_i \ \ and \ \ \sum_{i\in\mathcal{I}} ||f_i||^2 < \infty \right\},$$

with the inner product defined by

$$\langle \{f_i\}_{i\in\mathcal{I}}, \{g_i\}_{i\in\mathcal{I}} \rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i \rangle,$$

is a Hilbert space.

Definition 3.1. Let $W = \{W_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, $\{v_i\}_{i\in\mathcal{I}}$ be a family of weights, i.e. $v_i > 0$ and $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. We say $\Lambda := (\Lambda_i, W_i, v_i)$ is a generalized fusion frame (or g-fusion frame) for $\mathcal{H}$, if there exists $0 < A \leq B < \infty$ such that for each $f \in \mathcal{H}$

$$A||f||^2 \leq \sum_{i\in\mathcal{I}} v_i^2 ||\Lambda_i P_{W_i} f||^2 \leq B||f||^2. \tag{3}$$

We call Λ a Parseval g-fusion frame, if $A = B = 1$. When the right hand of (3) holds, Λ is called a g-fusion Bessel sequence for $\mathcal{H}$ with bound B. If $\mathcal{H}_i = \mathcal{H}$ for all $i \in \mathcal{I}$ and $\Lambda_i = I_{\mathcal{H}}$, then we get the fusion frame (W_i, v_i) for $\mathcal{H}$. Throughout this paper, Λ will be a triple (Λ_i, W_i, v_i) with $i \in \mathcal{I}$ unless otherwise stated.

Definition 3.2. Let Λ be a g-fusion frame for $\mathcal{H}$. Then, the analysis operator for Λ is defined by

$$U_\Lambda : \mathcal{H} \to \left(\sum_{i\in\mathcal{I}} \oplus \mathcal{H}_i\right)_{\ell^2}, \qquad U_\Lambda(f) = \{v_i \Lambda_i P_{W_i}(f)\}_{i\in\mathcal{I}}.$$

The adjoint of U_Λ is called the synthesis operator, we denote $T_\Lambda = U_\Lambda^*$.

By the elementary calculation, we have

$$T_\Lambda : \left(\sum_{i\in\mathcal{I}} \oplus \mathcal{H}_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_\Lambda(\{f_i\}_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} v_i P_{W_i} \Lambda_i^* f_i.$$

The g-fusion frame operator Λ is the composition of analysis and synthesis operators,

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \qquad S_\Lambda f = T_\Lambda U_\Lambda(f) = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} f.$$

We have

$$\langle S_\Lambda f, f\rangle = \sum_{i\in\mathcal{I}} v_i^2 ||\Lambda_i P_{W_i} f||^2.$$

Therefore

$$AI \leq S_\Lambda \leq BI.$$

This means that S_Λ is bounded, positive and invertible operator (with adjoint inverse). So, we have the reconstruction formula for any $f \in \mathcal{H}$

$$f = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} S_\Lambda^{-1} f = \sum_{i\in\mathcal{I}} v_i^2 S_\Lambda^{-1} P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} f.$$

The following theorem give the equivalence conditions between the g-fusion frames and their operators.

Theorem 3.3. *[5] Let Λ be the triple (Λ_i, W_i, v_i) with $i \in \mathcal{I}$. Then the following conditions are equivalent:*

(i) Λ is a g-fusion frame for $\mathcal{H}$,

(ii) The synthesis operator T_Λ is bounded, linear and onto,

(iii) The analysis operator S_Λ is well-defined, bounded, surjectire.

Lemma 3.4. *Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame with universal bounds A, B, and $Q \in B(\mathcal{H})$ be bounded below by $m > 0$, i.e.*
$m\|f\| \leq \|Qf\|$ for every $f \in \mathcal{H}$. Then $\Lambda = (\Lambda_i Q, W_i, v_i)$ is a g-fusion frame with bounds Am^2, and $B\|Q\|^2$.

Proof. for each $f \in \mathcal{H}$ we have

$$S_{\Lambda Q}f = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} Q^* \Lambda_i^* \Lambda_i Q P_{W_i} f = Q^*(\sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}) Q f = Q^*(S_\Lambda) Q f$$

where $S_\Lambda = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}$.
Since $m^2.I \leq Q^*Q, \quad A.I \leq S_\Lambda \leq B.I$, then

$$Am^2.I \leq A.Q^*Q \leq S_{\Lambda Q} \leq B.Q^*Q \leq B\|Q\|^2.I,$$

and we have the result. □

Corollary 3.5. *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a g-frame, and $Q \in B(\mathcal{H})$ be bounded below. Then $\{\Lambda_i Q\}_{i\in\mathcal{I}}$ is a g-frame.*

Corollary 3.6. *Let $\{W_i\}_{i\in\mathcal{I}}$ be a fusion frame family for $\mathcal{H}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$ and $Q \in B(\mathcal{H})$ be bounded below. Then $\{QW_i\}_{i\in\mathcal{I}}$ is a fusion frame with respect to $\{v_i\}_{i\in\mathcal{I}}$.*

[4, Proposition 2.1] leads us to the following result.

Theorem 3.7. *Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame. Let $V = \{V_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, and $\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. If $U : \mathcal{H} \to \mathcal{H}$ defined by*

$$U(f) = \sum_{i\in\mathcal{I}} v_i^2 (P_{V_i} \Gamma_i^* \Gamma_i P_{V_i}(f) - P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}(f)) \qquad (f \in \mathcal{H})$$

is a compact operator, then $\Gamma := (\Gamma_i, V_i, v_i)$ is g-fusion frame for $\overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$.

Proof. Let A and B be the g-fusion frame bounds for (Λ_i, W_i, v_i). we have

$$\|S_\Lambda\| = \sup_{\|f\|\leq 1} |\langle S_\Lambda f, f\rangle| = \sup_{\|f\|\leq 1} \sum_{i\in\mathcal{I}} v_i^2 ||\Lambda_i P_{W_i} f||^2 \leq B.$$

A simple calculation shows that U is a self-adjoint operator on $\mathcal{H}$. so if $T : \mathcal{H} \to \mathcal{H}$ is defined by $T = S_\Lambda + U$, then T is a bounded, linear, self-adjoint operator. Therefore we have

$$\|T\| = \sup_{\|f\|\leq 1} |\langle Tf, f\rangle| = \sup_{\|f\|\leq 1} \sum_{i\in\mathcal{I}} v_i^2 ||\Gamma_i P_{V_i} f||^2$$

and

$$\sum_{i\in\mathcal{I}} v_i^2 ||\Gamma_i P_{V_i} f||^2 \leq \|T\|\|f\|^2 \leq (\|S_\Lambda\| + \|U\|)\|f\|^2 \leq (B + \|U\|)\|f\|^2 \qquad (f \in \mathcal{H}) \qquad (*).$$

Now we obtain a lower bound for (Γ_i, V_i, v_i). Since U is a compact operator on $\mathcal{H}$, so US_Λ^{-1} is also a compact operator. Therefore the operator $US_\Lambda^{-1} + id_\mathcal{H}$ has closed range. As result $T = (US_\Lambda^{-1} + id_\mathcal{H})S_\Lambda$ is closed range. We consider T as an operator on the closed subspace $\overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$. Now we show that T is injective. If $f \in \overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$ and $T(f) = 0$, then

$$\sum_{i\in\mathcal{I}} v_i^2 ||\Gamma_i P_{V_i} f||^2 = \langle Tf, f\rangle = 0,$$

so $f = 0$. Also we have

$$Range(T) = (N(T^*))^\perp = (N(T))^\perp = \overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}.$$

Hence T is surjective and for every $f \in \overline{span}\{\Gamma_i^*(V_i)\}_{i\in\mathcal{I}}$ we have

$$\|T^{-1}\|^{-1}\|f\| \leq \|T(f)\| \leq \|T\|\|f\|.$$

Now by using the Cauchy-Schwartz inequality and $(*)$, we have

$$\begin{aligned}
\|T(f)\|^4 &= \big(\big\langle \sum_{i\in\mathcal{I}} v_i^2 P_{V_i}\Gamma_i^*\Gamma_i P_{V_i}(f), T(f)\big\rangle\big)^2 = \big(\big\langle \sum_{i\in\mathcal{I}} v_i^2 \Gamma_i P_{V_i}(f), \Gamma_i P_{V_i}(T(f))\big\rangle\big)^2 \\
&\leq \big(\sum_{i\in\mathcal{I}} v_i^2 \|\Gamma_i P_{V_i}(f)\|\|\Gamma_i P_{V_i}(T(f))\|\big)^2 \\
&\leq \big(\sum_{i\in\mathcal{I}} v_i^2 \|\Gamma_i P_{V_i}(f)\|^2\big)\big(\sum_{i\in\mathcal{I}} v_i^2 \|\Gamma_i P_{V_i}(T(f))\|^2\big) \\
&\leq (B + \|U\|)\|Tf\|^2\big(\sum_{i\in\mathcal{I}} v_i^2 \|\Gamma_i P_{V_i}(f)\|^2\big).
\end{aligned}$$

Therefore

$$\|Tf\|^2 \leq (B + \|U\|)\big(\sum_{i\in\mathcal{I}} v_i^2 \|\Gamma_i P_{V_i}(f)\|^2\big),$$

so

$$\sum_{i\in\mathcal{I}} v_i^2 \|\Gamma_i P_{V_i}(f)\|^2 \geq (B + \|U\|)^{-1}\|Tf\|^2 \geq (B + \|U\|)^{-1}\|T^{-1}\|^{-2}\|f\|^2.$$

□

4 Conclusion

We conclud that if $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame and $Q \in B(\mathcal{H})$ be bounded below by $m > 0$, Then $\Lambda = (\Lambda_i Q, W_i, v_i)$ is a g-fusion frame. Also if $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a g-frame, and $Q \in B(\mathcal{H})$ be bounded below. Then $\{\Lambda_i Q\}_{i\in\mathcal{I}}$ is a g-frame.

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