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Tridiagonal 3-Toeplitz Matrices

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Abstract

This paper gives an answer by presenting of eigenvalues for real tridiagonal 3-Toeplitz matrices of different order. It surveys the central results of the theory by finding roots of a combination of Chebyshev polynomials of the second kind. This answer solves the eigenproblem for integer powers of such matrices.

Keywords: 3-Toeplitz matrix, Chebyshev Polynomials, Eigenvalue

Mathematics Subject Classification [2010]: 15A03, 15A18, 65F15

1 Introduction

Toeplitz matrices are frequently used in many fields of science and engineering such as solving the inverse of a matrix, systems of linear equations, problems in the field of sound propagation, the stability of difference approximations to differential equations, etc. k-Toeplitz matrices are tridiagonal matrices of the form $A = [a_{i,j}]_{i,j=1}^{n}$ (with $n \geq k$) such that $a_{i+k,k+j} = a_{i,j} \quad (i, j = 1, 2, ..., n-k)$, so that they are k-periodic along the diagonals parallel to the main diagonal.
A Toeplitz matrix is a k-Toeplitz matrix when $k = 1$.
In this paper, we focus on the case of the form

$$A_n = \begin{pmatrix} a_1 & b_1 & & & & & & \\ c_1 & a_2 & b_2 & & & & & \\ & c_2 & a_3 & b_3 & & & & \\ & & c_3 & a_1 & b_1 & & & \\ & & & c_1 & a_2 & b_2 & & \\ & & & & c_2 & a_3 & b_3 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{n \times n}. \tag{1}$$

This matrix is an $n \times n$ real tridiagonal 3-Toeplitz matrices. The description of some explicit expressions for eigenvalues of a tridiagonal 3-Toeplitz matrices is the main topic of this note.

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2 Main results

The first key idea of our work is the expressions for eigenvalues of a tridiagonal 3-Toeplitz matrices by the following theorem:

Theorem 2.1. *(Marcella'n and Petronilho [4]). Let A_n, $n = 3, 4, 5, \ldots$, be the irreducible tridiagonal 3-Toeplitz matrix given by (1), where b_1, b_2, b_3, c_1, c_2 and c_3 are positive numbers. Define the sequence $\{S_n\}_{n\geq 0}$ of orthogonal polynomials associated with the matrices A_n as*

$$S_{3k}(x) = (b_1b_2b_3)^{-k}\{P_k(\pi_3(x)) + b_3c_3(x-a_2)P_{k-1}(\pi_3(x))\}, \tag{2}$$

$$S_{3k+1}(x) = b_1^{-1}(b_1b_2b_3)^{-k}\{(x-a_1)P_k(\pi_3(x)) + b_1c_1b_3c_3P_{k-1}(\pi_3(x))\}, \tag{3}$$

$$S_{3k+2}(x) = (b_1b_2)^{-1}(b_1b_2b_3)^{-k}(x-\xi_1)(x-\xi_2)P_k(\pi_3(x)),\ k = 0, 1, \ldots, \tag{4}$$

where ξ_1 and ξ_2 are the roots of the polynomial

$$(x-a_1)(x-a_2) - b_1c_1, \tag{5}$$

and

$$\pi_3(x) := \begin{vmatrix} x-a_1 & 1 & 1 \\ b_1c_1 & x-a_2 & 1 \\ b_3c_3 & b_2c_2 & x-a_3 \end{vmatrix}. \tag{6}$$

Then the eigenvalues $\lambda_{n,m}$ of A_n are the zeros of S_n, and the corresponding eigenvectors $\mathbf{v_{n,m}}$ are given by

$$\mathbf{v_{n,m}} = \begin{pmatrix} S_0(\lambda_{n,m}) \\ S_1(\lambda_{n,m}) \\ \vdots \\ S_{n-1}(\lambda_{n,m}) \end{pmatrix}, \quad m = 1, 2, \ldots, n. \tag{7}$$

Define

$$P_n(x) = (b_1b_2b_3c_1c_2c_3)^{n/2}U_n\left(\frac{x - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right),\ n = 0, 1, 2, \ldots, \tag{8}$$

where $U_n(x)$ is the Chebyshev polynomial of degree n of the second kind with
$n \in \mathbb{N} \cup \{-1, 0\}$.
All Chebyshev polynomials, among them $U_n(x)$, satisfy the three-term recurrence relations [3]:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad (U_{-1}(x) = 0,\ U_0(x) = 1,\ U_1(x) = 2x).$$

The sequence $\{S_k\}_k$ is an orthogonal polynomial sequence corresponding to a positive definite case. So, the zeros are simple and interlace [3, 5], i.e., if $\{x_{k,j}\}_{j=1}^{k}$ denotes the zeros of the polynomial S_k, then $x_{k,j} < x_{k-1,j} < x_{k,j+1}$, $j = 1, 2, ..., k-1$.
When $n = 3k+2$, from Equation (4), the eigenvalues $\lambda_{3k+2,m}$ of A_{3k+2} $(m = 1, 2, ..., 3k+2)$ are $\lambda_{3k+2,1} = \xi_1$, $\lambda_{3k+2,2} = \xi_2$ in the solutions of the cubic equations

$$Q(\lambda) := \pi_3(\lambda) - \left[b_1c_1 + b_2c_2 + b_3c_3 + 2\sqrt{b_1b_2b_3c_1c_2c_3}\cos\frac{i\pi}{k+1}\right] = 0,\ i = 1, \ldots, k. \tag{9}$$

From (6)

$$\begin{aligned}\pi_3(\lambda) = {} & (\lambda-a_1)(\lambda-a_2)(\lambda-a_2) - (b_1c_1 + b_2c_2 + b_3c_3)(\lambda-a_3) \\ & + b_2c_2(a_1-a_3) + b_3c_3(a_2-a_3) + b_1c_1 + b_2c_2 + b_3c_3,\end{aligned} \tag{10}$$

and from Shengjin formulas are given in [6], we compute the roots of the cubic Equation (9). The coefficients $Q(\lambda)$ of Equation (9) are

$q_1 = 1,\ q_2 = -(a_1 + a_2 + a_3),\ q_3 = a_1a_2 + a_2a_3 + a_1a_3 - b_1c_1 - b_2c_2 - b_3c_3,$
$q_4 = a_3b_1c_1 + a_1b_2c_2 + a_2b_3c_3 - a_1a_2a_3 - 2\sqrt{b_1b_2b_3c_1c_2c_3}\cos\frac{i\pi}{k+1}.$

Let

$\Delta_1 = q_2^2 - 3q_1q_3,\ \Delta_2 = q_2q_3 - 9q_1q_4,\ \Delta_3 = q_3^2 - 3q_2q_4,\ \Delta_4 = \Delta_2^2 - 4\Delta_1\Delta_3.$

Then we have

(1) If $\Delta_1 = \Delta_2 = 0$, $Q(\lambda)$ has only one real triple root;
(2) If $\Delta > 0$, $Q(\lambda)$ has one real root and a pair of conjugate imaginary roots;
(3) If $\Delta = 0$, $Q(\lambda)$ has three real roots: one simple and the other double;
(4) If $\Delta < 0$, $Q(\lambda)$ has three different real roots.

The corresponding eigenvectors $\mathbf{v_{n,m}}$ are given by (7).

When $n = 3k+1$, in Equation (3), the eigenvalues $\lambda_{3k+1,m}$ of A_{3k+1} $(m = 1, 2, ..., 3k+1)$ are the roots x of $S_{3k+1}(x)$ satisfy the equation

$$b_1^{-1}(b_1b_2b_3)^{-k}\{(x-a_1)P_k(\pi_3(x)) + b_1c_1b_3c_3P_{k-1}(\pi_3(x))\} = 0. \tag{11}$$

With following (8) in Equation (11), we have $s = \frac{\sqrt{b_1b_3c_1c_3}}{\sqrt{b_2c_2}}$.
If x is not a common root of $U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)$ and $a_1 - x$, then we conclude

$$\frac{U_n\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)}{U_{n-1}\left(\frac{\pi_3(x)-b_1c_1-b_2c_2-b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)} = \frac{s}{a_1 - x}. \tag{12}$$

Note 1. Let $\eta_0 < \xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{i-1} < \xi_i < \eta_i < \xi_{i+1} < \ldots < \eta_{n-1} < \xi_n < \eta_n$ with $\eta_0 = -\infty,\ \eta_n = \infty$, where $\xi_1,\ \xi_2, \ldots,\ \xi_n$ are the roots of $U_n(x)$, and $\eta_1,\ \eta_2, \ldots,\ \eta_{n-1}$ are the roots of $U_{n-1}(x)$ in Equation (12). Let $U_n(x)/U_{n-1}(x) = p_{n,n-1}(x)$, $n \geq 1$ and $p_{0,-1}(x) = 1$. Next we denote $g(x) = s/(a_1 - x)$ that here $s = \frac{\sqrt{b_1b_3c_1c_3}}{\sqrt{b_2c_2}}$.

Theorem 2.2. *If $s > 0$, for some i in Equation (12) and Note 1, then there are precisely two additional roots, exactly one lying in each of the intervals*

$$(\eta_{i-1}, a_1) \text{ and } (a_1, \eta_i).$$

If $s < 0$, then one or two additional roots of Equation (12) can be zero, in the interval (η_{i-1}, η_i). Finally, the next elseif $s = 0$, the problem is solved easily by finding roots of $U_n(x)$.

Note that, here $s > 0$. Then by the results of Theorem 2.2, the function (12) has the same roots as

$$h(x) \equiv (a_1 - x)U_n\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right) - sU_{n-1}\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right), \tag{13}$$

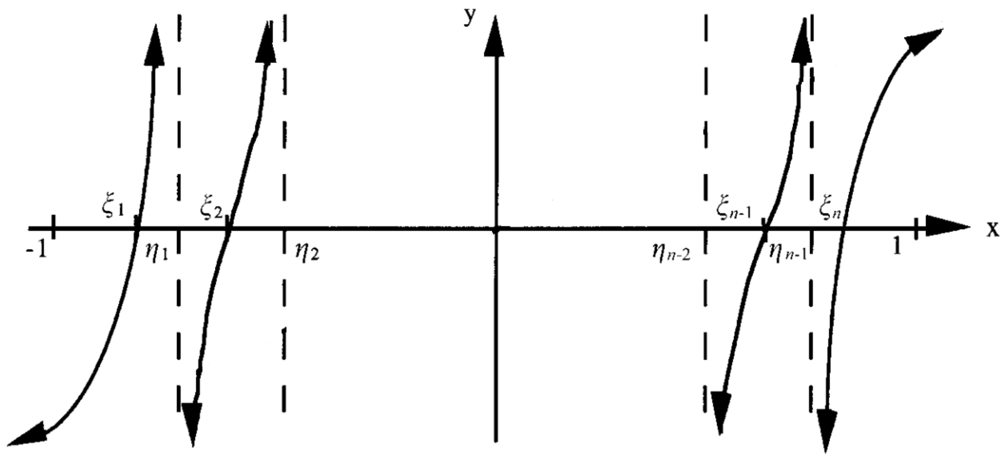

Figure 1: $p_{n,n-1}(x)$

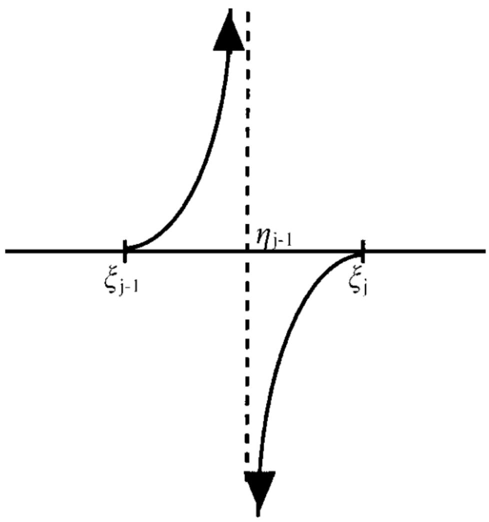

Figure 2: $p_{n,n-1}(x)$

The graph of $p_{n,n-1}(x)$ is shown in Fig. 1. Also Fig. 2 shows $p_{n,n-1}(x)$ in the interval (ξ_{j-1}, ξ_j).
For more details see [5].
Now, we have the roots of the following function by to approximate function $h(x)$ by Chebyshev interpolation for every interval (η_{i-1}, η_i), $i = 1, \ldots, n$, then use Chebyshev companion matrix to find roots.
To increase the accuracy, we can apply Chebfun for this work [3]. Chebfun finds roots with a global rootfinding capability by a method that shows in [5].

When $n = 3k$, from Equation (2), the eigenvalues $\lambda_{3k,m}$ of A_{3k} $(m = 1, 2, ..., 3k)$ are the roots x of $S_{3k}(x)$ satisfy the equation

$$(b_1 b_2 b_3)^{-k}\{P_k(\pi_3(x)) + b_3 c_3 (x - a_2) P_{k-1}(\pi_3(x))\} = 0. \tag{14}$$

If x is not a common root of $U_n\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)$ and $a_2 - x$, then we conclude

$$\frac{U_{n-1}\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)}{U_n\left(\frac{\pi_3(x) - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right)} = \frac{\sqrt{b_1b_2c_1c_2}}{\sqrt{b_3c_3}(a_2 - x)}. \tag{15}$$

Here, suppose $U_{n-1}(x)/U_n(x) = p_{n-1,n}(x)$, $n \geq 1$ and $g(x) = s/(a_2 - x)$ that $s = \frac{\sqrt{b_1b_2c_1c_2}}{\sqrt{b_3c_3}}$.
Where $\xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{i-1} < \xi_i < \eta_i < \xi_{i+1} < \ldots < \eta_{n-1} < \xi_n$.
$\xi_1,\ \xi_2, \ldots,\ \xi_n$ are the roots of $U_n(x)$ and $\eta_1,\ \eta_2, \ldots,\ \eta_{n-1}$ are the roots of $U_{n-1}(x)$.
Now, we apply Theorem 2.2 for finding eigenvalues of matrix (1) when $n = 3k$.

We show another way to the problem concerning the study of the eigenvalues of the sequences of matrices defined by (1), based on some results in [1,2]. We will study the case when the order $n = 3k$ of the matrix A_n in (1). Then A_n is the block Toeplitz matrix

$$A_n = \begin{pmatrix} B_0 & B_1 & & & & & \\ B_{-1} & B_0 & B_1 & & & & \\ & B_{-1} & B_0 & B_1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & B_1 \\ & & & & & B_{-1} & B_0 \end{pmatrix},$$

generated by the 3×3 matrix valued polynomial

$$f(x) := B_0 + B_1 e^{ix} + B_{-1} e^{-ix}$$

with

$$B_0 = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix}, \quad B_{-1} = \begin{pmatrix} 0 & 0 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, from Theorem 2.1, we know $b_1,\ b_2,\ b_3,\ c_1, c_2$ and c_3 are positive numbers and so it is well-known that, under such conditions, A_n is similar to the block Toeplitz matrix $\hat{A}_n$ by diagonal transformations, that is generated by the 3×3 matrix valued polynomial

$$\hat{f}(x) := \hat{B}_0 + \hat{B}_1 e^{ix} + \hat{B}_{-1} e^{-ix}$$

with

$$\hat{B}_0 = \begin{pmatrix} a_1 & \sqrt{b_1c_1} & 0 \\ \sqrt{b_1c_1} & a_2 & \sqrt{b_2c_2} \\ 0 & \sqrt{b_2c_2} & a_3 \end{pmatrix}, \quad \hat{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{b_3c_3} & 0 & 0 \end{pmatrix}, \quad \hat{B}_{-1} = \begin{pmatrix} 0 & 0 & \sqrt{b_3c_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are some papers for the Evaluation of the Eigenvalues of a Banded Toeplitz Block Matrix, such as [1, 2].

3 Conclusion

In this note we have considered a novel analysis review of spectral problem involving a tridiagonal 3-Toeplitz matrix for the cases $n = 3k+2$, $n = 3k+1$ and $n = 3k$ with some details on explicitly or implicitly tools.

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