The 11th International Seminar on
Linear Algebra and its Applications
Feb 2022

# Some topological properties of nonlinear positive mappings between $C^*$-algebras

Ali Dadkhah*

[1]Department of Mathematics, Ferdowsi University of Mashhad, Iran

**Abstract**

It is known that every positive linear map between unital $C^*$-algebras is norm-norm continuous. However, this is not necessarily true in the setting of nonlinear positive maps between $C^*$-algebras, in general. In this talk, we present some conditions to provide the continuity of nonlinear positive maps between $C^*$-algebras with respect to the strong and norm operator topologies. These results imply the automatic continuity of a large family of nonlinear positive maps between $C^*$-algebras. Moreover, we show that in some classes of nonlinear positive maps the continuity property implies the operator monotonicity.

**Keywords:** nonlinear positive map; $n$-monotone map; continuity; strong operator topology.
2020 *Mathematics Subject Classification*: 47A63, 47A08, 15A60.

## 1 Introduction

Let us denote by $\mathbb{B}(\mathscr{H})$ the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. In the case when $\mathscr{H}=\mathbb{C}^n$, we identify $\mathbb{B}(\mathbb{C}^n)$ with the matrix algebra of $n\times n$ complex matrices $M_n(\mathbb{C})$. Here we consider the usual Löwner order $\leq$ on the real space of self-adjoint operators. An operator $A$ is said to be strictly positive (denoted by $A>0$) if it is a positive invertible operator. Due to the Gelfand–Naimark–Segal theorem, we may assume that any $C^*$-algebra is a closed $C^*$-subalgebra of $\mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. We use $\mathscr{A},\mathscr{B},\cdots$ to denote $C^*$-algebras and $\mathscr{A}_+$ and $\mathscr{A}_{++}$ to denote the sets of all positive and positive invertible elements of $\mathscr{A}$, respectively. To shorten notation, we denote the relative strong operator weak operator topologies on any subset of $\mathbb{B}(\mathscr{H})$ by SOT and WOT, respectively.
Let $\Phi:\mathscr{A}\to\mathscr{B}$ be a (not necessarily linear) map between $C^*$-algebras. Then $\Phi$ is said to be $*$-map or self-adjoint if it is $*$-preserving i.e. $\Phi(A^*)=\Phi(A)^*$ and it is called positive if $\Phi$ maps the cone of positive elements of $\mathscr{A}$ to $\mathscr{B}_+$. If $\Phi(\mathscr{A}_{++})\subset\mathscr{B}_{++}$, then $\Phi$ is called strictly positive. We say that $\Phi$ is unital if $\mathscr{A},\mathscr{B}$ are unital and $\Phi$ preserves the unit. We simply denote both units of $\mathscr{A}$ and $\mathscr{B}$ by $I$. A map $\Phi$ is called $n$-positive if the map $\Phi_n: M_n(\mathscr{A})\to M_n(\mathscr{B})$ defined by $\Phi_n([a_{ij}])=[\Phi(a_{ij})]$ is positive, where $M_n(\mathscr{A})$ is the $C^*$-algebra of $n\times n$ matrices with entries in $\mathscr{A}$. If $\Phi$ is $n$-positive for all $n\in\mathbb{N}$, then it is called a completely positive map. We say a positive map $\Phi:\mathscr{A}\to\mathscr{B}$ is $n$-momotone, whenever the map $\Phi_n$ is monotone on $M_n(\mathscr{A})_+$ in the sense that if $[A_{ij}],[B_{ij}]\in M_n(\mathscr{A})_+$, then

$$[A_{ij}]\geq[B_{ij}]\Longrightarrow\Phi_n([A_{ij}])\geq\Phi_n([B_{ij}]).$$

*Email address: dadkhah61@yahoo.com

## 2 Main results

We start our work by giving some examples of $n$-positive maps and $n$-monotone maps.

**Example 2.1.** 1. If $\mathscr{A}$ is a $C^*$-algebra, then the operator norm of $\mathscr{A}$ is a 2-positive map from $\mathscr{A}$ to $\mathbb{C}$, which is not 2-monotone in general.

2. Every $2n$-positive map is $n$-monotone.

3. For every $1 \leq p < 2$, the power function $\Phi_p : \mathbb{C} \to \mathbb{C}$ defined by $\Phi_p(x) = |x|^p$ is 2-monotone. Note that $\Phi_p$ is 3-positive but not 4-positive.

4. Every positive semidefinite matrix $P \in M_n(\mathbb{C})$ induces a map $\phi_P : M_n(\mathbb{C}) \to \mathbb{C}$ defined by $\phi_P(A) = |\mathrm{tr}(AP)|$, which is a 3-positive semi–norm on $M_n(\mathbb{C})$. Moreover, $\phi$ is 2-monotone.

5. Every positive linear functional $\varphi : \mathscr{A} \longrightarrow \mathbb{C}$ on a $C^*$-algebra induces a nonlinear 3-positive and 2-monotone map $\Phi : \mathscr{A} \longrightarrow \mathbb{C}$ given by $\Phi(A) = |\varphi(A)|$.

6. The positive map $\Psi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ given by

$$\Psi(X) = \begin{cases} X, & X \neq -I, \\ 0, & X = -I, \end{cases}$$

is a 2-positive and 1-monotone map. However, $\Phi$ is neither 3-positive nor 2-monotone.

In the study of the continuity (with respect to various topologies) of a map between $C^*$-algebras a natural question is, what is the contribution of the positivity and the linearity? The exitance of several example of linear maps which are discontinuous is konwn. It is easy to present an example of discontinuous positive map. Indeed, consider the map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{C}$ given by

$$\Phi(X) = \begin{cases} \|X\| & \|X\| \leq I, \\ 0 & \text{otherwise}, \end{cases}.$$

Then $\Phi$ is a discontinuous unital positive map. Moreover, The map $\Psi$ (is defined in Example 2.1-6) is a discontinuous 2-positive map. Hence, the 2-positivity property is not enough to get the continuity of a map, in general. However, we can deduce a type of continuity with respect to SOT for 2-positive maps as follows.
If $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras, then a positive map $\Phi : \mathscr{A}_+ \to \mathscr{B}_+$ is called *normal* if for every bounded increasing net $\{A_\alpha\}_\alpha \subseteq \mathscr{A}_+$, it holds that

$$\Phi(\sup_\alpha A_\alpha) = \sup_\alpha \Phi(A_\alpha).$$

A map $\Phi : \mathscr{A}_+ \to \mathscr{B}_+$ is said to be *weakly normal* if for every norm convergent increasing net $\{A_\alpha\}_\alpha \subseteq \mathscr{A}_+$, the equlity

$$\Phi(\sup_\alpha A_\alpha) = \sup_\alpha \Phi(A_\alpha)$$

holds.
The first result reads as follows.

**Theorem 2.2.** *If $\Phi : \mathscr{A} \to \mathscr{B}$ is a 2-positive map between von Neumann algebras, then it is weakly normal.*

In [2, Example 2.3], authors show that if $\mathscr{H}$ is an infinite-dimensional separable Hilbert space, then the operator norm $\|\cdot\| : \mathbb{B}(\mathscr{H}) \to \mathbb{C}$ is an example of a 2-positive map, which is not normal.
We extend our work and give some results about the continuity of 3-positive maps.

**Theorem 2.3.** *Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a 3-positive map between unital $C^*$-algebras. Consider a topology $\tau \in \{\|\cdot\|, \mathrm{SOT}, \mathrm{WOT}\}$ on $\mathscr{A}$. Then the following conditions are equivalent:*

1. *$\Phi$ is norm-$\tau$-continuous,*
2. *$\Phi\left(\frac{1}{n}I\right) \xrightarrow{\tau} \Phi(0)$,*
3. *$\Phi\left(Z + \frac{1}{n}I\right) \xrightarrow{\tau} \Phi(Z)$ for some $Z \in \mathscr{A}_{++}$.*

Now we have the following result.

**Proposition 2.4.** *If $\Phi : \mathscr{A} \to \mathscr{B}$ is a 3-positive map between unital $C^*$-algebras, then*

$$\mathrm{SOT}-\lim_n \Phi\left(\frac{1}{n}I\right) = \Phi(0).$$

We notice that the norm continuity of the sequence $\{\Phi\left(\frac{1}{n}I\right)\}$ is still an open problem. According to Theorem 2.3 and Proposition 2.4, we get the following corollary.

**Corollary 2.5.** *All 3-positive maps between unital $C^*$-algebras are norm-SOT continuous. In addition, if a 3-positive map is norm-norm continuous at a positive invertable operator, then it is norm-norm continuous on its domain.*

The map $\Psi$ (is defined in Example 2.1-6) is a 2-positive map which is not norm-SOT continuous. Hence, the 3-positivity condition in the above results is necessary.
Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a 3-positive map between unital $C^*$-algebras. In the case, when $\mathscr{B}$ is finite-dimensional, Corollary 2.5 ensures that $\Phi$ is norm continuous. In particular, every 3-positive map $\Phi : \mathscr{A} \to M_n(\mathbb{C})$ is norm-continuous.
Now we aim to investigate the continuity of $n$-monotone mappings between $C^*$-algebras. The next result shows that the 2-monotonicity provides suitable conditions to ensure the continuity of a nonlinear positive map.

**Theorem 2.6.** *[2, Theorem 3.2] If $\Phi : \mathscr{A} \to \mathscr{B}$ is a 2-monotone map between unital $C^*$-algebras, then $\Phi$ is norm-continuous.*

The map $\Psi$ (is defined in Example 2.1-6) is a 1-monotone map which is not norm continuous. Hence, the 2-monotonicity condition in Theorem 2.6 is necessary.
It is known that every 4-positive map is 2-monotone; see [2, Section 4]. Therefore, all 4-positive maps between unital $C^*$-algebras are norm-continuous.
We aim to prove that in some cases a continuity condition for positive maps provides a monotonicity property for them.

**Definition 2.7.** [1, Definition IX.5.6] A map $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ is called a Lieb function if it satisfies the following two conditions:

- (Monotonicity) $\varphi(A) \geq \varphi(B) \geq 0$ if $A \geq B \geq 0$;
- (Cauchy–Schwarz) $\varphi(A^*A)\varphi(B^*B) \geq |\varphi(A^*B)|^2$ for every $A, B \in M_n(\mathbb{C})$.

An extension of the above definition was presented and investigated in [3] as follows:

**Definition 2.8.** A map $\Phi : \mathscr{A} \to \mathscr{B}$ between $C^*$-algebras is said to be a Lieb map if it has the following properties:

- (Monotonicity) $\Phi(A) \geq \Phi(B)$ if $A \geq B \geq 0$;
- (Cauchy–Schwarz) $\begin{bmatrix} \Phi(A^*A) & \Phi(A^*B) \\ \Phi(A^*B)^* & \Phi(B^*B) \end{bmatrix} \geq 0$ for all $A, B \in \mathscr{A}$.

The final result reads as follows.

**Theorem 2.9.** *Let $\mathscr{A}$ be a von Neumann algebra and let $\mathscr{B}$ be a unital $C^*$-algebra. If $\Phi : \mathscr{A} \to \mathscr{B}$ is an* SOT-SOT*-continuous map, then the Cauchy–Schwarz condition in Definitions 2.7 and 2.8 imply the monotonicity conditions.*

**Corollary 2.10.** *In the definition of* SOT-SOT*-continuous Lieb maps as well as in the definition of continuous Lieb functions, the monotonicity condition is redundant.*

## 3 Conclusion

The automatic continuity of positive linear maps between unital $C^*$-algebras can be extended to the framwork of nonlinear positive mappings if they are $n$-positive or $m$-monotone for sufficiently large $n, m \in \mathbb{N}$. Moreover, it seems that the $n$-monotonicity provides suitable conditions to imply the continuity of a nonlinear positive map. In addition, in many cases the reverse of the above facts hold. More precisely, the $n$-positivity and $m$-monotonicity of nonlinear positive maps can be deduced from a continuity property and a Cauchy–Schwarz type condition for such maps.

## Acknowledgment

The author wishes to express his thanks to Professor M.S. Moslehian and Professor M. Kian for their effective collaboration in reasearch on nonlinear positive maps.

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