MAPS PRESERVING THE λ-LIE PRODUCT OF OPERATORS

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Abstract. Let $\mathcal{A}$ be a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. In this paper, we characterize the forms of additive maps on $\mathcal{A}$ which strongly preserve the square zero of λ-Lie product of operators, i.e., if $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map which satisfies

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number λ with $\lambda \neq -1$, then it is shown that there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$, for every $A \in \mathcal{A}$.

1. Introduction

In last decade, Many mathematician research on the preserving problems. Specially, maps preserving a certain property of products of elements are considered. We point to some of them close to our purpose.

Let $\mathcal{A}$ be a Banach algebra, $A, B \in \mathcal{A}$ and λ be a scalar. $AB + \lambda BA$ is said to be the λ-Lie product of A and B and is denoted by $[A, B]_\lambda$. λ-Lie product is said to be Jordan product and Lie product, whenever $\lambda = 1$ and $\lambda = -1$, respectively. Lie product is denoted by $[A, B]$, too. Moreover, triple Jordan product A and B is defined as ABA. These products play rather important role in mathematical physics.

In [3], authors consider the maps strongly preserve the η-Lie product, that is $\phi(A)\phi(P) + \eta\phi(P)\phi(A) = AP + \eta PA$, for every A, some idempotent P and some scalar η.

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Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. Recall that a standard operator algebra on X is a norm closed subalgebra of $\mathcal{B}(\mathcal{X})$ which contains the identity and all finite rank operators. In [2], authors characterize the form of unital surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of product of operators, in both directions. Also in [4], authors characterize the form of linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of either products of operators or triple Jordan products of operators. In [1], Authors characterize the form of linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of Jordan products of operators.

We say that a map $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ strongly preserves the square zero of λ-Lie product of operators whenever

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$.

In this paper, we characterize the forms of additive maps which strongly preserve the square zero of λ-Lie products of operators. Our main result is the following theorem.

Theorem 1.1. *Assume that $\mathcal{A}$ is a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. Let $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ be an additive map which satisfies*

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number λ with $\lambda \neq -1$. Then there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$, for every $A \in \mathcal{A}$.

2. Proofs

First we recall some notations. We denote by $\mathcal{X}^*$, the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. We denote by $\mathcal{F}_1(\mathcal{X})$ the set of all rank one operators in $\mathcal{B}(\mathcal{X})$. The rank one operator $x \otimes f$ is idempotent if and only if $f(x) = 1$ and is nilpotent if and only if $f(x) = 0$.

To prove the main theorem, we must first prove the following results.

Proposition 2.1. *Let $A \in \mathcal{A}$, $x \in \mathcal{X}$, $f \in \mathcal{X}^*$ such that $f(x) \neq 0$ and $\lambda \neq 0, -1$. Then $[A, x \otimes f]_\lambda^2 = 0$ if and only if one of the following statements occurs:*

(i) $Axf(Ax) = -\lambda x f(A^2 x)$ and $Axf(x) = -\lambda x f(Ax)$.

(ii) $fA = 0$.

In the following lemmas, assume $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is a map which satisfies

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number λ with $\lambda \neq 0, -1$.

Lemma 2.2. $\ker A \subseteq \ker \phi(A)$*, for every $A \in \mathcal{A}$.*

Next assume that ϕ is additive.

Lemma 2.3. *$\phi(A) = 0$ or $\phi(A) = \kappa(A)A$, for every rank one operator A, where $\kappa : \mathcal{A} \to \mathbb{C}$ is a function.*

References

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