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Numerical simulation of a Coronavirus disease model with using an efficient nonstandard finite difference scheme

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Abstract

In this manuscript, we introduce a new nonstandard finite difference (NSFD) scheme to approximate solution of the coronavirus disease 2019 (COVID–19) model. In the beginning, the positivity and boundedness of solution of the COVID–19 model are discussed. The stability analysis of the equilibrium points the proposed COVID–19 model are then analyzed. Lastly, to ascertain the efficacy and accuracy of the suggested NSFD scheme, some numerical results are provided.

Keywords: COVID–19 model, Nonstandard finite difference scheme, Positivity, Boundedness, Stability.

Mathematics Subject Classification [2010]: 34D05, 92D30.

1 Introduction

Mathematical modelling plays a basic role in predicting and controlling present and future epidemics. Some patients with pneumonia of unidentified cause appeared in some medical institutions in December 2019 which happened in China. The World Health Organization (WHO) has announced the outbreak COVID–19 as a pandemic on March 2019. As of the end of April 2020, more than 2 millions COVID–19 cases and 200 thousand deaths have been reported from more than 200 countries. Medicine is continuously evolving in terms of refining, revising and discovering new knowledge about COVID–19. To bock the spread of the virus, there are some strategies such as citywide lock down, traffic halt, community management and social distance that have been adapted by the governments some countries in the world. In many cases, mathematical modelling of the COVID–19 can be described by a nonlinear system of ordinary differential equations (ODEs), (see [1, 3] for more details). A very few numbers of nonlinear ODEs can be solved by an analytical solution. Most of these ODEs cannot be solved by the well-known analytical method suitably. For this reason, various numerical methods were discussed to solve such ODEs. In this work, in order to approximate the solution of the COVID–19 model, we will

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construct an efficient NSFD scheme which is positive. A sensible model for the COVID–19 at time t can be described by the following initial value problem

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = A - dS - \beta SI - \gamma, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - dI, \\ \frac{\mathrm{d}R}{\mathrm{d}t} = \gamma - dR, \\ S(0) = S_0, \ I(0) = I_0, \ R(0) = R_0. \end{cases} \tag{1}$$

In this model, $S(t)$ is the number of the susceptible individuals at time t, $I(t)$ stands the infective individuals at time t and $R(t)$ denotes the recovered individuals at time t. Here, the death rates of the susceptible, the infective and the recovered are the same which is denoted by d. Also, the birth rate of the susceptible is A and the susceptible become the infective at a rate βSI where β is the contact rate. Moreover, the susceptible individuals become the recovered by the constant rate γ which is assumed that $A > \gamma$. This paper is structured as follows. Positivity and boundedness the solution of model (1) are proved in Section 2. Stability analysis of the equilibrium points of proposed COVID–19 model are investigated in Section 3. Section 4, is devoted to the study of an efficient NSFD scheme for the numerical solution of proposed COVID–19 model. Finally, numerical results are given in Section 5.

2 Positivity and boundedness

In this part, we want to prove positivity and boundedness of the solution model (1).

Theorem 2.1. *If $A > \gamma$ and $S(0),\ I(0),\ R(0) > 0$, then for all $t \geq 0$, $S(t),\ I(t)$ and $R(t) > 0$.*

Proof. Since the SR-coordinate plane is invariant under the flows of system, hence for all $t \geq 0$, $I(t) > 0$. Let $C = \{t \geq 0 | S(t) < 0\}$ and $D = \{t \geq 0 | R(t) < 0\}$. We will show that $C = \emptyset$. Suppose that $C \neq \emptyset$ and $C_0 = \inf(C)$, therefore $S(C_0) = 0$. Since $S(0) > 0$, thus $C_0 > 0$. By assumption, $C_0 = \inf(C)$ it follows that $S(t) \geq 0$, for all $t \in [0, C_0]$. This implies that from the third equation of the system (1), $S'(C_0) = A - \gamma > 0$. Hence, there exists $\varepsilon > 0$, such that $S'(t) > 0$, for all $t \in (C_0 - \varepsilon,\ C_0 + \varepsilon)$. Therefore, for all $t \in (C_0, C_0 + \varepsilon)$, $S(t) > S(C_0) = 0$, which contradicts $C_0 = \inf(C)$. By a similar argument, we can show that $R(t) \geq 0$, for all $t \geq 0$. □

In order to prove the boundedness of solution model (1), we first state the following proposition.

Proposition 2.2. *Let $K(t) : [0, +\infty) \longrightarrow \mathbb{R}$ be a derivative function such that $K(t) \geq 0$ for all $t \geq 0$. If $\alpha > 0$, $\beta \in \mathbb{R}$, such that $K'(t) + \alpha K(t) \leq \beta$, for every $t \geq 0$, then $K(t) \leq K(0) + \frac{\beta}{\alpha}$.*

Lemma 2.3. *If $A \geq \gamma$, then for all $t \geq 0$, we have $S(t) + I(t) + R(t) \leq S(0) + I(0) + R(0) + \frac{A}{d}$.*

Proof. Define $K(t) = \frac{1}{A}(S(t) + I(t) + R(t))$, hence $K'(t) + dK(t) \leq 1$. It follows from Proposition 2.2 that $K(t) \geq K(0) + \frac{1}{d}$. This establishes the desired result. □

3 Stability analysis of the COVID–19 model

The equilibrium points of the model (1) are given by $E_1 = (\frac{A-\gamma}{d}, 0, \frac{\gamma}{d})$, $E_2 = (\frac{d}{\beta}, \frac{A - \frac{d^2}{\beta} - \gamma}{d}, \frac{\gamma}{d})$.

Theorem 3.1. *The system (1) is*

(i) *locally asymptotically stable around* E_1 *if* $\gamma < A < \frac{d^2}{\beta}$.

(ii) *locally asymptotically stable around* E_2 *if* $\frac{\beta}{d}(A-\gamma) - d > 0$.

Proof. The variational matrix of system (1) corresponding to any arbitrary equilibrium point (S^*, I^*, R^*) can be expressed as

$$J(S^*, I^*, R^*) = \begin{pmatrix} -d-\beta I^* & -\beta S^* & 0 \\ \beta I^* & \beta S^* - d & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

At equilibrium point E_1, The variational matrix is

$$J(E_1) = \begin{pmatrix} -d & \frac{-\beta}{d}(A-\gamma) & 0 \\ 0 & \frac{\beta}{d}(A-\gamma) - d & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = -d$, $\lambda_2 = \frac{\beta}{d}(A-\gamma) - d$ and $\lambda_3 = -d$. Therefore, the equilibrium point E_1 is locally asymptotically stable if and only if $\frac{\beta}{d}(A-\gamma) - d < 0$. At the equilibrium point E_2, The variational matrix is

$$J(E_2) = \begin{pmatrix} -d-\beta I_1 & -d & 0 \\ \beta I_1 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix},$$

where $I_1 = \frac{1}{d}(A - \frac{d^2}{\beta} - \gamma)$. The characteristic equation the above matrix is $P(\lambda) = (\lambda + d)(\lambda^2 + (d+\beta I_1)\lambda + d\beta I_1)$. One eigenvalue of the above Jacobian matrix is $\lambda_1 = -d$. Observe that if $I_1 > 0$ then all of roots the polynomial $P^*(\lambda) = \lambda^2 + (d+\beta I_1)\lambda + d\beta I_1$, are negative. Hence the equilibrium point E_2 is locally asymptotically stable. $\square$

4 A new NSFD scheme for the COVID–19 model

The NSFD schemes were firstly introduced by Mickens. In order to introduce the general aspect of a NSFD scheme consider the following initial value problem

$$X'(t) = f(X(t)), \quad X(t_0) = X_0. \tag{2}$$

Suppose that a discretization $t_k = kh$, is given. A NSFD scheme for the problem (2) is constructed by the following two steps (see for instance [2]).

(i) The first order derivative in the initial value problem (2) at time step $t = t_k$ is replaced by a discrete form $X'(t_k) \approx \frac{X_{k+1} - X_k}{\phi(h)}$, where X_k is an approximation of $X(t_k)$ and the denominator function $\phi(h)$ satisfies the condition $\phi(h) = h + O(h^2)$ with $0 < \phi(h) < 1$.

(ii) The linear and nonlinear terms in the initial value problem (2) can be replaced by nonlocal discrete approximations.

Based on the Mickens rules, a NSFD scheme for the COVID–19 model (1) can be written as

$$\begin{cases} \dfrac{S_{k+1} - S_k}{\phi} = A - \gamma - dS_{k+1} - \beta S_{k+1} I_k, \\ \dfrac{I_{k+1} - I_k}{\phi} = \beta S_{k+1} I_k - dI, \\ \dfrac{R_{k+1} - R_k}{\phi} = \gamma - dR_{k+1}, \end{cases} \tag{3}$$

where $\phi(h) = \frac{e^{dh}-1}{d}$. A simple computation shows that

$$\begin{cases} S_{k+1} = \dfrac{(A-\gamma)\phi + S_k}{1 + d\phi + \beta\phi I_k}, \\ I_{k+1} = \dfrac{(1+\beta\phi S_k)I_k}{1+d\phi}, \\ R_{k+1} = \dfrac{\gamma\phi + R_k}{1+d\phi}. \end{cases} \tag{4}$$

Proposition 4.1. *If S_0, I_0 and $R_0 > 0$, then for all stepsize h, the values S_k, I_k and R_k are always positive.*

5 Simulation results

In this part, the numerical solutions of the proposed NSFD scheme on the two cases are presented. At the first simulation, we choose the parameter values $A = 0.5$, $d = 0.3$, $\beta = 0.5$ and $\gamma = 0.21$ with the initial condition $S_0 = 25$, $I_0 = 30$ and $R_0 = 20$ for simulating time 200 and stepsize $h = 0.4$. Figure 1 confirms that the NSFD scheme (4) converges to the equilibrium point $E = (0.6, 0.366, 0.7)$. In Figure 2, we plot the behaviour of the NSFD scheme (4) for the parameter values $A = 0.5$, $d = 0.7$, $\beta = 0.5$ and $\gamma = 0.21$ with choosing stepsize $h = 2$ and the initial condition $S_0 = 25$, $I_0 = 30$ and $R_0 = 20$. The Figure 2 shows that (S_k, I_k, R_k) approaches the equilibrium point $E = (0.41, 0, 0.3)$. The results show that the numerical solutions of the proposed NSFD schemes preserves the main properties of the COVID–19 model such as, positivity and stability, even for large stepsize h.

6 Concluding remarks

In this work, we studied an efficient NSFD scheme for numerical solutions for the COVID–19 model. We portrayed the simulation results in Figures 1–2, which indicate the new NSFD scheme preserved the positivity and stability properties of the COVID–19 model, even for choosing the large stepsize h. As a future research work, we can focus on the fractional–order COVID–19 model and obtain an efficient NSFD scheme which preserves the positivity and stability properties of the fractional–order COVID–19 model.

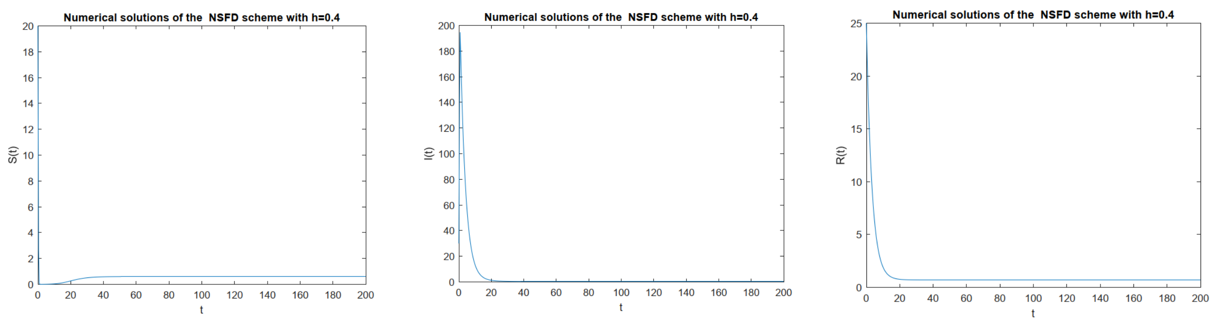

Figure 1: Numerical simulation with $h = 0.4$ for the NSFD scheme (3).

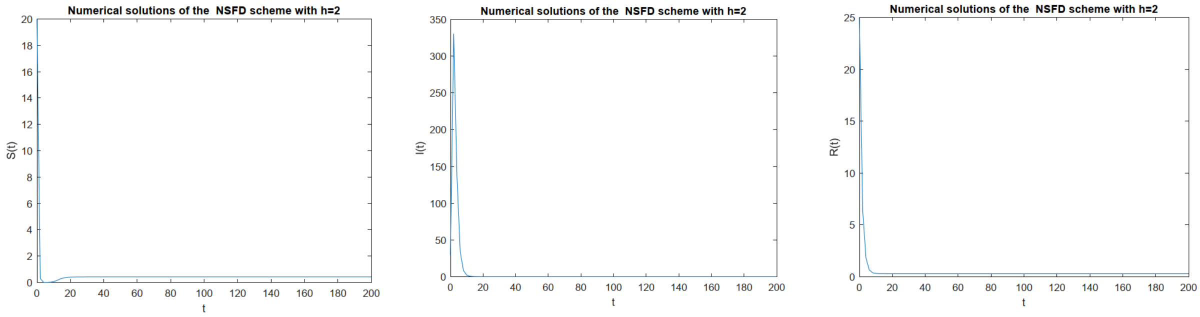

Figure 2: Numerical simulation with $h = 2$ for the NSFD scheme (3).

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