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On reduced minimum modulus preservers

Sepideh Haji Ghasemi[1*] and Shirin Hejazian[2]

[1,2]Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract

Suppose $\gamma(\cdot)$ denotes the reduced minimum modulus in a Banach algebra. We show that a continuous surjective linear map from a unital Banach algebra $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$ satisfying $\gamma(\varphi(a)\varphi(b)) = \gamma(ab)\ \ (a, b \in \mathcal{A})$, makes $\mathcal{A}$ to be a C^*-algebra $*$-isomorphic to $\mathcal{B}$ up to multiplication by a central invertible element. If $\mathfrak{B}(H)$ is the C^*-algebra of all bounded linear operators on a Hilbert space H, we characterize surjective maps φ on $\mathfrak{B}(H)$ (with no linearity and continuity assumption) preserving the reduced minimum modulus of operator products.

Keywords: reduced minimum modulus, preserver maps, operators on a Hilbert space

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1 Introduction

Throughout the paper all Banach spaces are assumed on the complex field. For a given Banach space X, $\mathcal{S}(X)$ and X^* denote the unit sphere of X and the dual of X, respectively, and $\mathfrak{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. For $T \in \mathfrak{B}(X)$, $R(T)$ and $\ker(T)$ denote the range and the null space of T, respectively. The unit circle in $\mathbb{C}$ will be denoted by $\mathbb{T}$. The *reduced minimum modulus* of an operator $T \in \mathfrak{B}(X)$ is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, \ker(T)) \geq 1\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0. \end{cases} \tag{1}$$

$T \in \mathfrak{B}(X)$, $\gamma(T) > 0$ if and only if $R(T)$ is closed (see [6, Part 10, Chapter II]). It is also defined for Banach algebra elements. Let $\mathcal{A}$ be a Banach algebra. For $a \in \mathcal{A}$, let L_a, R_a and $\sigma(a)$ denote the left and right multiplication operators by a and the spectrum of a, respectively. Harte and Mbekhta [5] considered the *left and right conorm* (*reduced minimum modulus*) of $a \in \mathcal{A}$ as follows

$$\gamma^{left}(a) = \gamma(L_a) = \inf\{\|ax\| : \operatorname{dist}(x, \ker(L_a)) \geq 1\},$$

where $\ker(L_a) = \{x \in \mathcal{A} : ax = 0\}$. Similarly,

$$\gamma^{right}(a) = \gamma(R_a) = \inf\{\|xa\| : \operatorname{dist}(x, \ker(R_a)) \geq 1\},$$

and $\ker(R_a) = \{x \in \mathcal{A} : xa = 0\}$. The *reduced minimum modulus* of $a \in \mathcal{A}$ is defined by $\gamma(a) := \gamma(L_a)$. If $\mathcal{A}$ is a C^*-algebra, $\gamma(L_a) = \gamma(R_a)$ (see [5, Theorem 4]). It is proved that if

*Speaker. Email address: sepide68ghasemi@gmail.com

a is invertible then $\gamma(a) = \|a^{-1}\|^{-1}$, see [5]. Suppose $\mathrm{Lan}(\mathcal{A})$ and $\mathrm{Ran}(\mathcal{A})$ denote the left and right annhilators of $\mathcal{A}$, respectively. Obviously, $a \in \mathrm{Lan}(\mathcal{A})$ if and only if $L_a = 0$ if and only if $\gamma(a) = +\infty$. Let $\mathcal{A}$ be a Banach algebra with a right (resp. left) approximate identity (e.g. a C^*-algebra), then $\mathrm{Lan}(\mathcal{A}) = \{0\}$ (resp. $\mathrm{Ran}(\mathcal{A}) = \{0\}$). So, for $a \in \mathcal{A}$, $\gamma(a) = +\infty$ if and only if $a = 0$. Also, if $\mathcal{A}$ is a C^*-algebra, then

$$\gamma(a)^2 = \inf\{\lambda : \lambda \in \sigma(a^*a) \setminus \{0\}\} \tag{2}$$

for all $a \in \mathcal{A}$, [5, Theorem 4]. Consequently, $\gamma(a) = \gamma(a^*a)^{\frac{1}{2}} = \gamma(aa^*)^{\frac{1}{2}} = \gamma(a^*)$. So, $\gamma(a)^2 = \gamma(a^2)$ whenever $a = a^*$. Moreover, if $u, v \in \mathcal{A}$ are unitary elements, then by definition $\gamma(uav) = \gamma(av)$ for all $a \in \mathcal{A}$. However, $\gamma(av) = \gamma(v^*a^*) = \gamma(a^*) = \gamma(a)$. Therefore, $\gamma(uav) = \gamma(a)$ for all $a \in \mathcal{A}$.

Let H be a Hilbert space. We denote by $\mathfrak{R}_1(H)$ the set of all rank one operators on H. We recall that every rank one operator T in $\mathfrak{B}(H)$ is of the form $T = x \otimes y$ for some $x, y \in H$, and $(x \otimes y)^* = y \otimes x$. So, $(x \otimes y)^*(x \otimes y) = (y \otimes x)(x \otimes y) = \|x\|^2 y \otimes y$. Thus, $\sigma((x \otimes y)^*(x \otimes y)) = \{0, \|x\|^2\|y\|^2\}$. Since $\mathfrak{B}(H)$ is a C^*-algebra, we have $\gamma(x \otimes y) = \|x\|\|y\|$.

We study surjective maps preserving the reduced minimum modulus of products. First, we assume that $\mathcal{A}$ is a unital Banach algebra *topologically generated by its idempotents*, that is $\mathcal{A}$ is the norm closure of its subalgebra generated by idempotents. If $\mathcal{B}$ is a C^*-algebra, and $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective continuous linear map satisfying $\gamma(ab) = \gamma(\varphi(a)\varphi(b))$ for all $a, b \in \mathcal{A}$, then we show that $\mathcal{B}$ is unital, $\varphi(1)$ is a central invertible element in $\mathcal{B}$ and $\varphi(1)^{-1}\varphi$ is an isomorphism. We also show that $\mathcal{A}$ is a C^*-algebra, $*$-isomorphic to $\mathcal{B}$. Then, we assume that H is a complex Hilbert space of dimension greater than 2 and study surjective maps (with no linearity and continuity assumption) on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of operator products. We show that a surjective map ϕ on $\mathfrak{B}(H)$ preserves the reduced minimum modulus of products if and only if ϕ is a linear or conjugate linear $*$-automorphism multiplied by a partial isometry. More precisely, $\phi(T) = U_T\psi(T) = \psi(T)V_T^*$ for all $T \in \mathfrak{B}(H)$, where ψ is a linear or conjugate linear $*$-automorphism and U_T, V_T are partial isometries on $\overline{R(\psi(T))}$ and $\overline{R(\psi(T)^*)}$, respectively. Finally, we study surjections on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of Jordan triple products of operators.

2 Preserving reduced minimum modulus of products on Banach algebras

If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras with $\mathrm{Lan}(\mathcal{B}) = \{0\}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ is a map satisfying

$$\gamma(\varphi(a)\varphi(b)) = \gamma(ab) \quad (a, b \in \mathcal{A}), \tag{3}$$

then obviously, for $a, b \in \mathcal{A}$, $ab = 0 \Rightarrow \varphi(a)\varphi(b) = 0$. Thus preserving zero product plays an important role in our arguments.

Lemma 2.1. *Let $\mathcal{A}$ be a unital Banach algebra which is topologically generated by its idempotents. Suppose $\mathcal{B}$ is a C^*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective continuous linear map preserving zero products in both directions. Then,*

(i) *φ is injective.*

(ii) *$\mathcal{B}$ is unital, $\varphi(1)$ is a central invertible element and $\varphi(1)^{-1}\varphi$ is an isomorphism.*

Proof. It is proved that φ is injective and [2, Lemma 2.1 and Theorem 2.2] leads to (ii). □

Remark 2.2. We recall that if $\mathcal{A}$ is a Banach algebra and Φ is an isomorphism from $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$ (which is automatically continuous), then it follows from defining $a^* = \Phi^{-1}(\Phi(a)^*)$ and $\|a\|_0 = \|\Phi(a)\|$ $(a \in \mathcal{A})$ that $(\mathcal{A}, *, \|\cdot\|_0)$ is a C^*-algebra. Being isomorphic to $\mathcal{B}$, the Banach algebra $\mathcal{A}$ is semisimple and so $\|\cdot\|_0$ is equivalent to the original norm on $\mathcal{A}$, and the two norms coincide if and only if Φ is an isometry.

In the following theorem we show that if $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Lemma 2.1 and if $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous surjective linear map satisfying (3), then $\mathcal{A}$ is a C^*-algebra with respect to its original norm, which is $*$-isomorphic to $\mathcal{B}$.

Theorem 2.3. *Let $\mathcal{A}$ be a unital Banach algebra which is topologically generated by its idempotents. Suppose $\mathcal{B}$ is a C^*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective continuous linear map satisfying*

$$\gamma(\varphi(a)\varphi(b)) = \gamma(ab). \quad (a, b \in \mathcal{A})$$

Then, $\mathcal{A}$ is a C^*-algebra which is $*$-isomorphic to $\mathcal{B}$.

Proof. Since φ satisfies (3), it preserves zero products in both directions. So by Lemma 2.1, φ is injective, $\mathcal{B}$ is unital, $\varphi(1)$ is a central invertible element and $\Psi : \mathcal{A} \to \mathcal{B}$ defined by $\Psi(a) = \varphi(1)^{-1}\varphi(a)$ $(a \in \mathcal{A})$ is an isomorphism. By using [1, Lemma 4.1], one can see that $\mathcal{A}$ is a C^*-algebra and Ψ is a $*$-isomorphism. $\square$

3 Maps preserving reduced minimum modulus of operator products

Let H be a complex Hilbert space of dimension ≥ 3 and let $\mathcal{U}(H)$ denote the set of unitaries on H. In the sequel, we describe a surjective (with no linearity and continuity assumption) map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfying

$$\gamma\big(\phi(T)\phi(S)\big) = \gamma(TS) \quad (T, S \in \mathfrak{B}(H)). \tag{4}$$

As $\mathfrak{B}(H)$ is a C^*-algebra, (4) implies that ϕ preserves zero product. In addition, $\gamma(p) = \inf\{\lambda : \lambda \in \sigma(p^*p) \setminus \{0\}\}^{\frac{1}{2}} = 1$ for all projections $p \in \mathfrak{B}(H)$. Particularly, $\gamma(.)$ is constant on the set of all rank one projections. So, we have the same characterization as in [3, Theorem 2.3] on $\mathfrak{R}_1(H)$. Hence by a similar discussion leading to [3, Theorem 3.2], we see that a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (4) if and only if there exist a unitary or an anti-unitary U_0 in $\mathfrak{B}(H)$ and functions $h_1, h_2 : \mathfrak{B}(H) \to \mathcal{U}(H)$ satisfying $h_1(T)T = Th_2(T)$ for all $T \in \mathfrak{B}(H)$, such that

$$\phi(T) = U_0 h_1(T) T U_0{}^* = U_0 T h_2(T) U_0{}^*, \tag{5}$$

for all $T \in \mathfrak{B}(H)$.

Here by using properties of γ, we are going to find further necessary and sufficient conditions for ϕ to satisfy (4). To prove our main results we need the following Lemma.

Lemma 3.1. *Let $A, B \in \mathfrak{B}(H)$. Then the following statements are equivalent.*

(i) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{B}(H)$.

(ii) $\gamma(AT) = \gamma(BT)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii) $|A| = |B|$.

The following statements are also equivalent.

(i)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{B}(H)$.

(ii)′ $\gamma(TA) = \gamma(TB)$ *for all* $T \in \mathfrak{R}_1(H)$.

(iii)′ $|A^*| = |B^*|$.

We can state the following proposition by applying (5) and Lemma 3.1.

Proposition 3.2. *Let* H *be a complex Hilbert space with* $dimH \geq 3$, *and* $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ *a surjective map. Then* ϕ *satisfies* (4) *if and only if there exists a linear or conjugate linear* $*$*-automorphism* $\psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ *such that* $|\phi(T)| = |\psi(T)|$ *and* $|\phi(T)^*| = |\psi(T)^*|$ *for all* $T \in \mathfrak{B}(H)$.

The following Lemma and Proposition 3.2 leads to the next Theorem.

Lemma 3.3. *Let* $T \in \mathfrak{B}(H)$ *and* U *be a partial isometry on* $\overline{R(T)}$. *Then,* $\gamma(UTS) = \gamma(TS)$ *for all* $S \in \mathfrak{B}(H)$.

Theorem 3.4. *Let* H *be a complex Hilbert space with* $dimH \geq 3$, *and* $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ *a surjective map. Then* ϕ *satisfies* (4) *if and only if*

$$\phi(T) = U_T\psi(T) = \psi(T)V_T^* \quad (T \in \mathfrak{B}(H)),$$

where ψ *is a linear or conjugate linear* $*$*-automorphism on* $\mathfrak{B}(H)$ *and for each* $T \in \mathfrak{B}(H)$, U_T , V_T *are partial isometries on* $\overline{R(\psi(T))}$, $\overline{R(\psi(T)^*)}$, *respectively. As a consequence, there is a unitary or anti-unitary operator* U *on* H *such that* $\phi(T) = U_TUTU^* = UTU^*V_T^*$.

We recall that the Jordan triple product of operators T, S is TST. In the sequel, we consider a surjective maps ϕ on $\mathfrak{B}(H)$ satisfying

$$\gamma(\phi(T)\phi(S)\phi(T)) = \gamma(TST) \quad (T, S \in \mathfrak{B}(H)). \tag{6}$$

It is easily seen that such a map preserves zero Jordan triple product in both directions, that is

$$TST = 0 \Longleftrightarrow \phi(T)\phi(S)\phi(T) = 0. \tag{7}$$

We apply the characterization of maps satisfying (7) [4] to find a finer characterization for maps satisfying (6).

Remark 3.5. (1) Applying [4, Theorem 2.2], we conclude that if H is infinite dimensional and a surjection ϕ on $\mathfrak{B}(H)$ satisfies (7), then there is a function $\mu : \mathfrak{B}(H) \to \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator $A : H \to H$ such that either

(a) $\phi(T) = \mu(T)ATA^{-1} \quad (T \in \mathfrak{B}(H))$, or,

(b) $\phi(T) = \mu(T)AT^\star A^{-1} \quad (T \in \mathfrak{B}(H))$.

Here $T^\star$ denotes the Banach space adjoint of $T \in \mathfrak{B}(H)$. If J is the conjugate linear isomorphism from H onto its dual H^*, then it is easily seen that $T^\star = JT^*J^{-1}$, for all $T \in \mathfrak{B}(H)$. Therefore,

$$\phi(T) = \mu(T)AJT^*J^{-1}A^{-1} \quad (T \in \mathfrak{B}(H)).$$

Clearly, AJ is linear or conjugate linear depending on A is conjugate linear or linear, respectively. Renaming AJ into A, we arrive at

(b)′ $\phi(T) = \mu(T)AT^*A^{-1}$, for all $T \in \mathfrak{B}(H)$, where A is a linear or conjugate linear invertible operator.

(2) Suppose that $H = \mathbb{C}^n$, $n \geq 3$, and that $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a surjective map satisfying (7). Applying [4, Theorem 2.1] shows that there exist an invertible matrix $S \in M_n(\mathbb{C})$, a field automorphism $f_0 : \mathbb{C} \to \mathbb{C}$, and a scalar function $\mu : M_n(\mathbb{C}) \to \mathbb{C} \setminus \{0\}$ such that one of the following holds:

  (c) $\phi(A) = \mu(A)Sf(A)S^{-1} \quad (A \in M_n(\mathbb{C}))$, or

  (d) $\phi(A) = \mu(A)Sf(A)^{tr}S^{-1} \quad (A \in M_n(\mathbb{C}))$,
  where $f([a_{ij}]) = [f_0(a_{ij})]$.

In the two following theorems, we show that if a surjective map ϕ on $\mathfrak{B}(H)$ satisfies (6), then the invertible operators A and S in Remark 3.5(1)-(2) can be replaced by unitaries and moreover, $|\mu| = 1$. As a consequence, ϕ is norm preserving.

Theorem 3.6. *Let H be an infinite dimensional complex Hilbert space. A surjective map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ satisfies (6) if and only if there exist a function $\mu : \mathfrak{B}(H) \to \mathbb{T}$ and a unitary or anti-unitary operator U on H such that either $\phi(T) = \mu(T)UTU^*$ or $\phi(T) = \mu(T)UT^*U^*$, for all $T \in \mathfrak{B}(H)$.*

The proof of the following theorem follows the same line as the proof of [4, Theorem 4.1].

Theorem 3.7. *Suppose $n \geq 3$, then $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies (6) if and only if there exists a unitary matrix U and a function $\mu : M_n(\mathbb{C}) \to \mathbb{T}$ such that either*

(i) $\phi(A) = \mu(A)Uf(A)U^*$, *or*

(ii) $\phi(A) = \mu(A)U(f(A))^{tr}U^*$

for all $A = [a_{ij}] \in M_n(\mathbb{C})$. We have $f([a_{ij}]) = [f_0(a_{ij})]$ where, $f_0 : \mathbb{C} \to \mathbb{C}$ is the identity or the complex conjugate on $\mathbb{C}$.

4 Conclusion

A continuous linear map φ preserving the reduced minimum modules of products from a Banach algebra $\mathcal{A}$ topologically generated by its idempotents onto a C^*-algebra, makes $\mathcal{A}$ to be a C^*-algebra. Surjective maps on $\mathfrak{B}(H)$ preserving the reduced minimum modulus of products are $*$-isomorphisms multiplied by partial isometries and hence are isometries.

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