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A note on Maps preserving the local spectral subspace of skew-product of operators

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Abstract

Let $B(H)$ be the algebra of all bounded linear operators on infinite-dimensional complex Hilbert space H. For $T \in B(H)$ and $\lambda \in \mathbb{C}$, let $H_T(\{\lambda\})$ denotes the local spectral subspace of T associated with $\{\lambda\}$. We show that if an additive map $\varphi : B(H) \longrightarrow B(H)$ has a range containing all operators of rank at most two and satisfies

$$H_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\})$$

for all $T, S \in B(H)$ and $\lambda \in \mathbb{C}$, then there exist two unitary operators U and V in $B(H)$ such that $\varphi(T) = UTV^*$ for all $T \in B(H)$. Also, we obtain some interesting results in this direction.

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1 Introduction

Throughout this paper, H and K are two infinite-dimensional complex Hilbert spaces. As usual $B(H, K)$ denotes the space of all bounded linear operators from H into K. When $H = K$ we simply write $B(H)$ instead of $B(H, H)$, and its unit will be denoted by I. The inner product of H or K will be denoted by $\langle , \rangle$ if there is no confusion.
Linear preserver problems, in the most general setting, demand the characterization of linear maps between algebras that leave a certain property, a particular relation, or even a subset invariant. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [6], that determines linear maps preserving the determinant of matrices. The study of linear and nonlinear local spectra preserver problems attracted the attention of a number of authors. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [5] additive maps on the algebra of all linear bounded operators on a complex Banach space X that preserve the local spectrum of operators at each vector of X. Their results cleared the way for several authors to describe maps on matrices or operators that preserve local spectrum, local spectral radius, and local inner spectral radius; see, for instance, the last section of the survey article [3] and the references therein. Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space X and its unit will

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be denoted by I. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and a X-valued analytic function $f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. The local spectral radius of T at x is given by $r_T(x) := \limsup_{n\longrightarrow\infty} \|T^n(x)\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. We recall that an operator $T \in B(X)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) provided that for every open subset U of $\mathbb{C}$, the equation

$$(\mu I - T)f(\mu) = 0, \quad \forall \mu \in U,$$

has no nontrivial analytic solution f. Every operator $T \in B(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property.

For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $X_T(F)$ is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $X_T(F_1) \subseteq X_T(F_2)$. The book by P. Aiena [1] provide an excellent exposition as well as a rich bibliography of the local spectral theory.

In [2], H. Benbouziane et al. characterized the forms of surjective weakly continuous maps φ from $B(X)$ into $B(X)$ which satisfy

$$X_{\varphi(T)-\varphi(S)}(\{\lambda\}) = X_{T-S}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

In this paper, we investigate the form of all maps φ on $B(H)$ such that, for every T and S in $B(H)$, the local spectral subspaces of TS^* and $\varphi(T)\varphi(S)^*$ are the same associated with the singleton $\{\lambda\}$.

The first lemma summarizes some known basic properties of the local spectrum.

Lemma 1.1. *[1] Let X be a Banach space and $T \in B(X)$. For every $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*
(a) $\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.
(b) If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$.
(c) If $S \in B(X)$ commutes with T, then $\sigma_T(Sx) \subseteq \sigma_T(x)$.
(d) $\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ for all $x \in X$ and $n \in \mathbb{N}$.

In the next theorem we collect some of the basic properties of the subspaces $X_T(F)$.

Lemma 1.2. *[1] Let $T \in B(X)$ and $F \subseteq \mathbb{C}$. The following statements hold.*
(i) $X_T(F)$ is a T-hyperinvariant subspace of X.
(ii) $(T - \lambda I)X_T(F) = X_T(F)$ for every $\lambda \in \mathbb{C}\backslash F$.
(iii) $X_T(F) = X_T(F \cap \sigma(T))$.
(iv) If $x \in X$ satisfy $(T - \lambda I)x \in X_T(F)$, then $x \in X_T(F)$.
(v) $ker(T - \lambda I) \subseteq X_T(F)$.
(vi) $X_{\alpha T}(\lambda) = X_T(\frac{\lambda}{\alpha})$ for every $\lambda \in \mathbb{C}$ and non-zero scalar α.

The nonzero local spectrum of $T \in B(H)$ at any $x \in H$ is defined by

$$\sigma_T^*(x) := \begin{cases} \{0\} & if\ \sigma_T(x) = \{0\}, \\ \sigma_T(T) \setminus \{0\} & if\ \sigma_T(x) \neq \{0\}. \end{cases}$$

For any $x, y \in H$, let $x \otimes y$ denote the operator of rank at most one on H defined by

$$(x \otimes y)z = \langle z, y \rangle\, x, \qquad \forall\ z \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $T \in B(H)$ can be written as a finite sum of rank one operators; i.e., $T = \sum_{i=1}^{n} x_i \otimes y_i$ for some $x_i, y_i \in X$ and $i = 1, 2, ..., n$. We denote by $F(H)$ the set of all finite rank operators in $B(H)$ and $F_n(H)$ the set of all operators of rank at most n, n is a positive integer.

The following lemma is an elementary observation that gives the nonzero local spectrum of any rank one operator.

Lemma 1.3. *(See [4]) Let x_0 be a nonzero vector in H. For any $x, y \in H$, we have*

$$\sigma^*_{x\otimes y}(x_0) := \begin{cases} \{0\} & if\ \langle x_0, y\rangle = 0, \\ \langle x, y\rangle & if\ \langle x_0, y\rangle \neq 0. \end{cases}$$

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators in term of local spectrum.

Theorem 1.4. *(See [4, Theorem 4.1]) For a nonzero vector $x \in H$ and a nonzero operator $R \in B(H)$, the following statements are equivalent.*
(a) R has rank one.
(b) $\sigma^*_{RT}(x)$ contains at most one element for all $T \in B(H)$.
(c) $\sigma^*_{RT}(x)$ contains at most one element for every rank two operator $T \in B(H)$.

The following result characterizes in term of the local spectrum when two operators are the same.

Lemma 1.5. *(See [4, Theorem 3.2]) For a nonzero vector x in H and two operators A and B in $B(H)$, the following statements are equivalent.*
(a) $A = B$.
(b) $\sigma_{AT}(x) = \sigma_{BT}(x)$ for all operators $T \in B(H)$.
(c) $\sigma_{AT}(x) = \sigma_{BT}(x)$ for all rank one operators $T \in B(H)$.
(d) $\sigma^*_{AT}(x) = \sigma^*_{BT}(x)$ for all rank one operators $T \in B(H)$.

2 Main results

We first establish the following lemma.

Lemma 2.1. *Let $T, S \in B(H)$. The following statements are equivalent.*
(1) *$T = S$.*
(2) *$H_{RT^*}(\{\lambda\}) = H_{RS^*}(\{\lambda\})$ for all $\lambda \in \mathbb{C}$ and $R \in F_1(H)$.*

Proof. We only need to establish implication (2) ⇒ (1). □

Proposition 2.2. *If φ_1 and φ_2 are maps from $B(H)$ into $B(H)$ satisfy*

$$H_{\varphi_1(T)\varphi_2(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}),$$

then the following statements hold.
(i) φ_2 is injective.
(ii) If the range of φ_1 is contains $F_2(H)$ then φ_2 is homogeneous.

Theorem 2.3. *Let $\varphi : B(H) \to B(H)$ be an additive map such that its range contains $F_2(H)$. If*

$$H_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C})$$

then there exist two unitary operators U and V in $B(H)$ such that such that $\varphi(T) = UTV^*$ for all $T \in B(H)$.

Proof. The proof breaks down into several claims.

Claim 1. φ is injective and linear.

Claim 2. φ preserves rank one operators in both directions.

Claim 3. There are bijective linear mappings $A : H \to H$ and $B : H \to H$ such that $\varphi(x \otimes y) = Ax \otimes By$ for all $x, y \in H$.

Claim 4. A and B are bounded unitary operators multiplied by positive scalars α and β such that $\alpha\beta = 1$.

Claim 5. A^* and I are linearly dependent.

Claim 6. φ has the asserted form.

□

From this result, it is easy to deduce a generalization to the case of two different Banach spaces H, K.

Corollary 2.4. *Suppose $P \in B(H, K)$ be a unitary operator. Let φ be an additive map from $B(H)$ onto $B(K)$ which satisfy*

$$K_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = PH_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}).$$

Then there exists a unitary operator $Q : K \to H$ such that $\varphi(T) = PTQ$ for all $T \in B(H)$.

Corollary 2.5. *Let $P \in B(H, K)$ be an unitary operator. Let $\varphi : B(H) \to B(H, K)$ be an additive surjective map which satisfy*

$$K_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = PH_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}).$$

Then there exists an unitary operator $Q : H \to H$ such that $\varphi(T) = PTQ^*$ for all $T \in B(X)$.

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