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Generalized inverse of matrices

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Abstract

In this paper, some relations between Drazin, Moore-Penrose inverses with *DMP*, *CMP* generalized inverses are studied.

Keywords: Moore Penrose inverse, *Drazin inverse*, *CMP inverse*, *DMP inverse*

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1 Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices and assume A^*, $\mathcal{R}(A)$ and rank(A) denote the conjugate transpose, column space and rank of $A \in \mathbb{C}^{m\times n}$, respectively. For $A \in \mathbb{C}^{n\times n}$, the smallest nonnegative integer m defined by the condition rank(A^m)=rank(A^{m+1}) is called the index of A and is denoted by ind(A).
The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^D = X \in \mathbb{C}^{n\times n}$ such that

$$A^{m+1}X = A^m, \quad XAX = X, \quad AX = XA,$$

where $m = ind(A)$. If $ind(A) = 1$, then A^D is the group inverse of A, which is denoted by $A^{\#}$. The basic theory and various applications of the Drazin inverse can be found in the monographs [1,6].
The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger = X \in \mathbb{C}^{n\times m}$ which satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = XA, \quad (XA)^* = XA.$$

If X satisfies the equation $AXA = A$, then X is called a g-inverse of A. A matrix X is an outer inverse of A, if $XAX = X$ holds. An important feature of the Moore-Penrose inverse is that it can be used to represent orthogonal projectors. For instance, $P_A = AA^\dagger$ and $Q_A = A^\dagger A$ are the orthogonal projectors onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

In [5, Theorem 2.2.21] it has been proven that A can be written as the sum of two matrices A_1 and A_2 i.e. $A = A_1 + A_2$, where

- $rank(A_1) = rank(A_1^2)$ i.e. ind$(A_1) \leq 1$,
- A_2 is nilpotent,

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- $A_1A_2 = A_2A_1 = 0$.

The matrices A_1 and A_2 are called the core and nilpotent parts of A respectively, and this decomposition is unique.
The concept of *DMP inverse* of A was introduced in [4]. In this case, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X, \quad XA = A^DA, \quad and \quad A^mX = A^mA^\dagger,$$

is called the *DMP inverse* of A and is denoted by $A^{D,\dagger}$. Moreover, it was proved that $A^{D,\dagger} = A^DAA^\dagger$. Dually, it is easy to see that the dual *DMP inverse* is given by $A^{\dagger,D} = A^\dagger AA^D$.
The *CMP inverse* for a complex matrix was introduced by [2]. The *CMP inverse* of A is a matrix $X \in \mathbb{C}^{n\times n}$ such that the following equations hold:

$$XAX = X, \quad AXA = A_1, \quad AX = A_1A^\dagger, \quad XA = A^\dagger A_1.$$

Such matrix X is unique and denoted by $A^{c\dagger} = A^\dagger A_1 A^\dagger$.

2 Main results

In this section some relations between Drazin, inverses with *DMP*, *CMP* generalized inverses are studied. Easy calculations show the following lemma which will be helpful throughout the paper.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and let $A = A_1 + A_2$ be the core-nilpotent decomposition. Then*

1. $A^DA_1 = A_1A^D$,
2. A^D *is a g-inverse of* A_1,
3. A^D *is an outer inverse of* A.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$. Then $X = A^D$ is a solution of the following equations.*

$$A^{c,\dagger}XA^{c,\dagger} = A^{c,\dagger}XA^{D,\dagger} = A^{\dagger,D}XA^{D,\dagger} = A^\dagger A^D A^\dagger.$$

Proof. 1. We have

$$\begin{aligned}
A^{c,\dagger}XA^{c,\dagger} &= A^{c,\dagger}A^DA^\dagger AA^DAA^\dagger = A^{c,\dagger}(A^D)^2AA^\dagger AA^DAA^\dagger \\
&= A^{c,\dagger}(A^D)^2AA^DAA^\dagger = A^{c,\dagger}A^DA^DAA^\dagger = A^{c,\dagger}A^DA^{D,\dagger} \\
&= A^\dagger AA^DAA^\dagger A^DA^{D,\dagger} = A^\dagger AA^DAA^\dagger A(A^D)^2A^{D,\dagger} \\
&= A^\dagger AA^DA(A^D)^2A^{D,\dagger} = A^\dagger AA^DA^DA^{D,\dagger} = A^{\dagger,D}A^DA^{D,\dagger} \\
&= A^\dagger AA^DA^DA^DAA^\dagger = A^\dagger A^DA^DAA^\dagger = A^\dagger A^DA^\dagger
\end{aligned}$$

□

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$. Then $X = A^D$ is a solution of the following equations.*

$$A^{D,\dagger}XA^{\dagger,D} = X^2A^{D,\dagger}AX = X^2A^{\dagger,D} = (A^D)^3.$$

The *Hartwig-Spindelbock decomposition* [3, Corollary 6] of any matrix $A \in \mathbb{C}^{n\times n}$ of rank r is given by

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is a diagonal matrix of the nonzero singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = I_r.$$

The Drazin inverse and the Moore-penrose inverse of A are as follows [4].

$$A^D = U \begin{pmatrix} (\Sigma K)^D & ((\Sigma K)^D)^2\Sigma L \\ 0 & 0 \end{pmatrix} U^*, \quad \text{and} \quad A^\dagger = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*. \tag{2}$$

By using $A_1 = AA^DA$, we obtain the following

$$A_1 = U \begin{pmatrix} \Sigma K(\Sigma K)^D\Sigma K & \Sigma K(\Sigma K)^D\Sigma L \\ 0 & 0 \end{pmatrix} U^*. \tag{3}$$

By using Hartwig-Spindelbock decomposition, we obtain the following theorem.

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be as in (1). Then*

1. $A_1 = A^\dagger$ *if and only if* $(\Sigma K)_1 = K^*\Sigma^{-1}$ *and* $L = 0$,
2. *If* $A^D = A_1$ *is, then* $(\Sigma K)_1 = (\Sigma K)^D$,
3. $A_1 = A^{D,\dagger}$ *if and only if* $(\Sigma K)^D = \left((\Sigma K)^D\right)^3$ *and* $(\Sigma K)^D\Sigma L = 0$,
4. $A_1 = A$ *if and only if* $\Sigma K(\Sigma K)^D = I_r$,

   where $(\Sigma K)_1 = (\Sigma K)(\Sigma K)^D(\Sigma K)$.

Proof. 1. Let $A_1 = A^\dagger$. Then by using (2) and (3), we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = K^*\Sigma^{-1}, \qquad (\Sigma K)(\Sigma K)^D\Sigma L = 0, \qquad L^*\Sigma^{-1} = 0.$$

Therefore, $A_1 = A^\dagger$ if and only if $(\Sigma K)(\Sigma K)^D(\Sigma K) = K^*\Sigma^{-1}$ and $L = 0$.

2. Let $A^D = A_1$. Then by using (2) and (3), we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K)^D, \qquad (\Sigma K)(\Sigma K)^D(\Sigma L) = ((\Sigma K)^D)^2(\Sigma L). \tag{4}$$

The quation (4) is equivalent to the following equation

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = ((\Sigma K)^D)^2(\Sigma K), \qquad (\Sigma K)(\Sigma K)^D(\Sigma L) = ((\Sigma K)^D)^2(\Sigma L). \tag{5}$$

Right-multiplying both of these equalities (5) by K^*K and L^*K, respectively and using $KK^* + LL^* = I_r$, we get $(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K)^D$. Therefore $(\Sigma K)_1 = (\Sigma K)^D$.

3. Let $A_1 = A^{D,\dagger}$. Then by using (3) and [4, Theorem 2.5], we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K)^D, \qquad (\Sigma K)(\Sigma K)^D\Sigma L = 0. \tag{6}$$

Pre-multiplying both of these equalities (6) by $\left((\Sigma K)^D\right)^2$ and $(\Sigma K)^D$, respectively. Therefore, $A_1 = A^{D,\dagger}$ if and only if $(\Sigma K)^D = \left((\Sigma K)^D\right)^3$ and $(\Sigma K)^D \Sigma L = 0$.

4. Suppose that $A_1 = A$. Then by using (1) and (3), we have

$$(\Sigma K)(\Sigma K)^D(\Sigma K) = (\Sigma K), \qquad (\Sigma K)(\Sigma K)^D \Sigma L = \Sigma L. \tag{7}$$

Rigth-multiplying both of these equalities (7) by $K^*\Sigma^{-1}$ and $L^*\Sigma^{-1}$, respectively and using $KK^* + LL^* = I_r$, the equality $A_1 = A$ holds if and only if $\Sigma K(\Sigma K)^D = I_r$.

□

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ be as in (1) and $P_A = AA^\dagger$. Then*

(a) $P_A A_1 = A_1 P_A$ if and only if $(\Sigma K)^D \Sigma L = 0$,

(b) $A_1(I_n - P_A) = (I_n - P_A)A_1$ if and only if $(\Sigma K)^D \Sigma L = 0$,

(c) $A^{c\dagger}(I_n - P_A) = (I_n - P_A)A^{c\dagger}$ if and only if $L^*K(\Sigma K)^D = 0$.

Proof. (a). Let $A \in \mathbb{C}^{n\times n}$. By using (1), (2) and (3), we have

$$P_A A_1 = U \begin{pmatrix} \Sigma K(\Sigma K)^D \Sigma K & \Sigma K(\Sigma K)^D \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{8}$$

$$A_1 P_A = U \begin{pmatrix} \Sigma K(\Sigma K)^D \Sigma K & 0 \\ 0 & 0 \end{pmatrix} U^*. \tag{9}$$

By (8) and (9), the equality $P_A A_1 = A_1 P_A$ holds if and only if $\Sigma K(\Sigma K)^D \Sigma L = 0$.
Pre-multiplying the equation $\Sigma K(\Sigma K)^D \Sigma L = 0$ by $(\Sigma K)^D$ and using
$(\Sigma K)^D = (\Sigma K)^D(\Sigma K)(\Sigma K)^D$, we get $(\Sigma K)^D \Sigma L = 0$.
Therefore, $P_A A_1 = A_1 P_A$ if and only if $(\Sigma K)^D \Sigma L = 0$.
By (1), (2) and (3), (b) holds.
By [2, P.3(7)], (1) and (2), (c) holds.

□

3 Conclusion

In this paper, by using *DMP*, *CMP* generalized inverses, we obtain some equations. The relation between Drazin inverse and these equations are studied.

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