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Decomposition of Invariant Derivations as the Tensor Sum Form

Hamed Minaee Azari[1*] and Asadollah Niknam[2]

[1]Department of Pure Mathematics, Center Of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P. O. Box 1159,Mashhad 91775, Iran

[2]Department of Pure Mathematics, Center Of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P. O. Box 1159,Mashhad 91775, Iran

Abstract

Let A and B be C^*-algebras over a field $\mathbb{F}$, we show that every $\mathbb{F}$-invariant derivation δ of $A \otimes B$ can be represented as tensor sum $\delta = \Delta \otimes id + id \otimes \nabla$ where id stands for the identity operator, Δ and ∇ are derivations on A, B, respectively.

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1 Introduction

Let A and B are C^*-algebras over $\mathbb{F}$. If A, B do not have identities, denote by $A \oplus \mathbb{F}$, $B \oplus \mathbb{F}$ the C^*-algebras obtained by adjoining an identity to A, B, respectively. Since an arbitrary C^*-norm on $A \otimes B$ can be extended to a C^*-norm on $(A \oplus \mathbb{F}) \otimes (B \oplus \mathbb{F})$, the restriction of the spatial C^*-norm on $(A \oplus \mathbb{F}) \otimes (B \oplus \mathbb{F})$ to $A \otimes B$ is the spatial norm on $A \otimes B$. Thus we may assume that A and B have identity elements. The linear span of elements of the form $(x \otimes \mu)(\lambda \otimes y)$, where $x \in A$, $y \in B$, and $\mu, \lambda \in \mathbb{F}$, with $(x \otimes \mu)(\lambda \otimes y) = \lambda x \otimes \mu y$ is equal to $A \otimes B$. Then $A \otimes \mathbb{F}$ and $\mathbb{F} \otimes B$ are embeddable in the tensor product $A \otimes B$. Tensor sum of operators can be thought of as an extension to infinite dimensional spaces of the traditional Kronecker sum of matrices on finite dimensional spaces [1,3]. The goal of this paper is to prove that every invariant derivation of $A \otimes B$ is the tensor sum of the derivations. From the nature of this result, and the relative simplicity of its proof, one would expect that it is known; however, we have not been able to find it in the literature among related results. Let us now outline the contents of this paper. The main topic of the paper is contained in section 2, and we proved the main theorem about an invariant derivation on a tensor product. In the end of paper we explore corollares, examples and more results of tensor sum. Niknam [4] proved in 1993 the operator $\Delta \otimes id + id \otimes \nabla$ is a $*-$derivation on $A \otimes B$, where δ, ∇ are $*-$derivations over A and B, respectively. In this paper we show that if δ is a derivation on tensor product $A \otimes B$ with invariant property has the following form.

$$\Delta \otimes id + id \otimes \nabla.$$

*Speaker. Email address: minaeehamed@yahoo.com

In the next section we provide all lemmas and proof their. Let A be a C^*-algebra. Then a $*$-derivation δ of A means a linear mapping from A into A such that $\delta(xy) = \delta(x)y + x\delta(y)$, $\delta(x^*) = \delta(x)^*$, for every $x, y \in A$.
By $Der(A)$ we denote the set of all derivations of A. Further, for every $u \in A$ we define ad $u : A \longrightarrow A$ by ad $u(x) = ux - xu$. Note that ad $u \in Der(A)$ and such a derivation is said to be an inner derivation [4].
If A and B are normed spaces, and $u \in A \otimes B$, then there exist linearly independent sets $\{x_i\}$, $\{y_i\}$ such that $u = \sum_{i=1}^{n} x_i \otimes y_i$, see [2].

Lemma 1.1. *Let A be a normed space over a field $\mathbb{F}$. Then each element u of $A \otimes \mathbb{F}$ may be written uniquely in the form $u = x_u \otimes 1$, where $x_u \in A$. In particular, $\|u\| = \|x_u\|$.*

The following lemma will be needed in the proof of the main result.

Lemma 1.2. *Let A be C^*-algebra over a field $\mathbb{F}$, let δ be a $*$-derivation of $A \otimes \mathbb{F}$. Then there exists a $*$-derivation Δ of A such that for every $x \in A$ we have*

$$\delta(x \otimes 1) = \Delta(x) \otimes 1. \tag{1}$$

Furthermore, $\|\Delta\| \leq \|\delta\|$. Inparticular, if δ is closable, so is Δ.

A similar result same as the above lemma holds for a derivation δ of $\mathbb{F} \otimes B$, where B is a C^*-algebra over $\mathbb{F}$.
For example, if we put $y = 1$ in ad $u(x \otimes y) = \Delta(x) \otimes y$ for every $u \in A \otimes \mathbb{F}$, then ad $u = \Delta \otimes id$ is a derivation of $A \otimes \mathbb{F}$. If in addition δ is a derivation of $A \otimes \mathbb{F}$, then $\alpha\delta\alpha^{-1}$ is a derivation of A, where α is the isomorphism from $A \otimes \mathbb{F}$ onto A.

2 Main result

In this section, we peresent our main theorem. A mapping δ of tensor product $A \otimes B$ is called $\mathbb{F}$-invariant if $A \otimes \mathbb{F}$ and $\mathbb{F} \otimes B$ are invariant under δ [6].

Theorem 2.1. *Let A and B be C^*-algebras over $\mathbb{F}$. Then every $\mathbb{F}$-invariant derivation δ of $A \otimes B$ can be written as*

$$\delta = \Delta \otimes id + id \otimes \nabla,$$

where $\Delta \in Der(A)$, $\nabla \in Der(B)$.

3 corollaries

Corollary 3.1. *Let A, B be C^*-algebras. If every derivation of A and B are inner then every derivation of $A \otimes B$ is inner.*

An element z in $A \otimes B$ is said to be tensor sumable if there are x, y in A and B, respectively such that $z = x \otimes 1 + 1 \otimes y$.

Example 3.2.

The matrix $Z = \begin{pmatrix} 0 & 2 & 1 & 2 & 0 & 0 \\ -1 & 0 & 5 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & -1 & 0 & 3 \end{pmatrix}$ is an element of $M_{2\times 2}(\mathbb{R}) \otimes M_{3\times 3}(\mathbb{R})$, where $M_{n\times n}(\mathbb{R})$ is the set of all $n \times n$ matrices over $\mathbb{R}$, hear $n = 2, 3$. If $X = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 5 \\ -1 & 0 & 0 \end{pmatrix}$, then $Z = X \otimes I_3 + I_2 \otimes Y$ is a tensor sumable in $M_{2\times 2}(\mathbb{R}) \otimes M_{3\times 3}(\mathbb{R})$.

Corollary 3.3. *If A, B are non commutative, then every inner derivation ad z of $A \otimes B$ where z is a tensor sumable, can be written as tensor sum of inner derivations.*

Corollary 3.4. *Let A, B be C^*-algebras. If Δ and ∇ are inner derivations over, A, B, respectively. Then $\Delta \otimes id + id \otimes \nabla = adz$ for some tensor sumable $z \in A \otimes B$.*

A derivation δ on $A \otimes B$ is called tensor sumable if there exist two derivations Δ, ∇ over A and B, respectively such that $\delta = \Delta \otimes id + id \otimes \nabla$ and we write $\delta = \Delta \boxplus \nabla$. Also, the tensor difference of Δ, ∇ is denoted by $\Delta \boxminus \nabla$ and it is definition as follows:

$$\Delta \boxminus \nabla = \Delta \otimes id - id \otimes \nabla$$

Theorem 3.5. *Let A and B are C^*-algebras over $\mathbb{F}$. Then for every $\alpha, \beta \in \mathbb{F}$, $\Delta_1, \Delta_2 \in Der(A)$ and $\nabla_1, \nabla_2 \in Der(B)$,*
(i) $\alpha\beta(\beta^{-1}\Delta_1 \boxplus \nabla_1\alpha^{-1}) = \alpha\Delta_1 \boxplus \nabla_1\beta$ where α, β are non zero,
(ii) $\Delta_1 \boxplus \nabla_1 + \Delta_2 \boxplus \nabla_2 = \Delta_1 \boxplus \nabla_2 + \Delta_2 \boxplus \nabla_1$,
(iii) $\alpha(\Delta_1 \boxplus \nabla_1)\beta = \alpha\Delta_1 \boxplus \nabla_1\beta$,
(iv) $\Delta_1 \boxplus \Delta_1 = \Delta_1 \boxplus id + id \boxplus \Delta_1 - id \boxplus id$,
(v) $\Delta_1 \boxplus \nabla_1 = \Delta_1 \otimes \nabla_1$ iff $\Delta_1 \otimes id$ is a quasi-inverse of $id \otimes \nabla_1$,
(vi) $\|\Delta_1 \boxplus \nabla_1\| = \|\Delta_1\|\|\nabla_1\|$ if and only if $\Delta_1 \otimes id$ is a quasi-inverse of $id \otimes \Delta_1$,
(vii) If $\Delta \neq \lambda id$ for every non zero scalar $\lambda \in \mathbb{F}$, then $\Delta \boxplus \nabla \neq 0$,
$(viii)$ $-(\Delta_1 \boxplus \nabla_1) = -\Delta_1 \boxplus -\nabla_1$.
(ix) If Δ_1 and ∇_1 are normal, so is $\Delta \boxplus \nabla$.

Theorem 3.6. *Let Δ, ∇ be $*$-derivations on C^*-algebras A, B respectively. Then $\Delta \boxminus \nabla$ is a $*$-derivation on $A \otimes B$, and $(\Delta \boxplus i\nabla)^* = \Delta^* \boxminus i\nabla^*$. Furthermore,*

$$(\Delta \boxplus \nabla)(\Delta \boxminus \nabla) = \Delta^2 \boxminus \nabla^2$$

4 Conclusion

Let us suppose that δ be an $\mathbb{F}$-invariant derivation on $A \otimes B$ and restrictions of δ to $A \otimes \mathbb{F}$, $\mathbb{F} \otimes B$ be δ_1, δ_2, respectively. Since $A \otimes \mathbb{F}$ and $\mathbb{F} \otimes B$ are isomorphic to A, B, respectively, then there exist isomorphisms maps $\alpha : A \otimes \mathbb{F} \longrightarrow A$ given by $\alpha(x \otimes r) = rx$ and $\beta : \mathbb{F} \otimes B \longrightarrow B$ given by $\beta(s \otimes y) = sy$. Take $\Delta = \alpha\delta_1\alpha^{-1}$, and $\nabla = \beta\delta_2\beta^{-1}$, therefore $\delta = \Delta \otimes id + id \otimes \nabla$. then Δ and ∇ are derivation, for if $x, x' \in A$, then the liniearity of Δ follows immediately that $\alpha\delta_1\alpha^{-1}$ is a linear operator. It is enough to show that Δ satisfies Leibniz rule. To this

$$\Delta(xx') = (\alpha\delta_1\alpha^{-1})(xx') = (\alpha\delta_1)(\alpha^{-1}(x).\alpha^{-1}(x'))$$

$$
\begin{aligned}
&= \alpha(\delta_1(\alpha^{-1}(x))\alpha^{-1}(x') + \alpha^{-1}(x)\delta_1(\alpha^{-1}(x')))\\
&= \alpha(\delta_1(x \otimes 1)(x' \otimes 1) + (x \otimes 1)\delta_1(x' \otimes 1))\\
&= \alpha\delta_1(x \otimes 1)\alpha(x' \otimes 1) + \alpha(x \otimes 1)\alpha\delta_1(x' \otimes 1)\\
&= \alpha\delta_1\alpha^{-1}(x)x' + x\alpha\delta_1\alpha^{-1}(x')\\
&= \Delta(x)x' + x\Delta(x').
\end{aligned}
$$

Similarly, we can show that ∇ is a derivation on B. Now we prove $\delta = \Delta \otimes id + id \otimes \nabla$. Let $x \otimes y$ be an arbitrary elemen in $A \otimes B$ we have

$$
\begin{aligned}
(\Delta \otimes id + id \otimes \nabla)(x \otimes y) &= \Delta(x) \otimes y + x \otimes \nabla(y)\\
&= \alpha\delta_1\alpha^{-1}(x) \otimes y + x \otimes \beta\delta_2\beta^{-1}(y)\\
&= \alpha\delta_1(x \otimes 1) \otimes y + x \otimes \beta\delta_2(1 \otimes y)\\
&= \alpha(a \otimes 1) \otimes y + x \otimes \beta(1 \otimes b),
\end{aligned}
$$

where $\delta_1(x \otimes 1) = a \otimes 1$ and $\delta_2(1 \otimes y) = 1 \otimes b$ (such elements exist for invariance of δ). Hence

$$
\begin{aligned}
\alpha(a \otimes 1) \otimes y + x \otimes \beta(1 \otimes b) &= a \otimes y + x \otimes b\\
&= (a \otimes 1)(1 \otimes y) + (x \otimes 1)(1 \otimes b)\\
&= \delta(x \otimes 1)(1 \otimes y) + (x \otimes 1)\delta(1 \otimes y)\\
&= \delta((x \otimes 1)(1 \otimes y))\\
&= \delta(x \otimes y).
\end{aligned}
$$

Thus, $\delta = \Delta \otimes id + id \otimes \nabla$.

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