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# A martingale inequality in tracial von Neumann algebras

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**Abstract**

In this talk, using the Lieb–Araki concavity, we obtain a noncommutative version of Freedman's inequality for martingales, which gives an upper bound for the tail probabilities of a supermartingale in the setting of von Neumann algebras.

**Keywords:** Noncommutative probability space; trace; Freedman inequality; martingale.
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## 1 Introduction

Freedman's inequality [2] asserts that if $\{X_n : X_0 = 0, n \geq 0\}$ is a martingale in the classical probability space $(\mathbb{P}, \Omega, \mathcal{F})$ with martingale difference sequence $\{D_n : D_0 = 0, n \geq 0\}$ such that $D_n = X_n - X_{n-1} \leq 1$ for all $n$, then, for all $c \geq 0$ and $h > 0$, it holds that

$$\mathrm{Prob}\big(X_n \geq c \text{ and } Y_n \leq h \text{ for some } n \geq 0\big) \leq \left(\frac{h}{c+h}\right)^{c+h} e^c,$$

in which $Y_n = \sum_{k=1}^{n} \mathbb{E}_{k-1}(D_k^2)$ is predictable quadratic variation, where $\mathbb{E}_{k-1}$ denotes the conditional expectation onto the $k-1$'st $\sigma$-algebra in the underlying filter. Applying the Lieb concavity theotem, an extension of Freedman's inequality is provided by Tropp [6] to the case of matrix martingales.

In this talk, we are inspired by some ideas in the commutative case and Tropp's result to provide a Freedman-type inequality in the framework of noncommutative probability spaces.

A von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathcal{H}$ with unit element 1 equipped with a normal faithful tracial state $\tau : \mathcal{M} \to \mathbb{C}$ is called a noncommutative probability space. We denote by $\leq$ the usual order on the self-adjoint part $\mathcal{M}^{sa}$ of $\mathcal{M}$. For each self-adjoint operator $a \in \mathcal{M}$, there exists a unique spectral measure $E$ as a $\sigma$-additive mapping with respect to the strong operator topology from the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ into the set of all orthogonal projections such that for every Borel function $f : \sigma(a) \to \mathbb{C}$ the operator $f(a)$ is defined by $f(a) = \int f(\lambda) dE(\lambda)$, in which $\sigma(a)$ is the spectrum of $a$, in particular, $\mathbf{1}_B(x) = \int_B dE(\lambda) = E(B)$.

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then the conditional expectation $\mathcal{E}_{\mathcal{N}}$ of $\mathcal{M}$ with respect to $\mathcal{N}$ is a normal positive contractive projection $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ satisfying the following properties:
(i) $\mathcal{E}_{\mathcal{N}}(axb) = a\mathcal{E}_{\mathcal{N}}(x)b$ for any $x \in \mathcal{M}$ and $a, b \in \mathcal{N}$;

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(ii) $\tau \circ \mathcal{E}_{\mathcal{N}} = \tau$.
Moreover, $\mathcal{E}_{\mathcal{N}}$ is the unique mapping satisfying (i) and (ii).

A filtration of $\mathcal{M}$ is an increasing sequence $(\mathcal{M}_j, \mathcal{E}_j)_{0\leq j\leq n}$ of von Neumann subalgebras of $\mathcal{M}$ together with the conditional expectations $\mathcal{E}_j$ of $\mathcal{M}$ with respect to $\mathcal{M}_j$ such that $\bigcup_j \mathcal{M}_j$ is $w^*$–dense in $\mathcal{M}$.

A sequence $(a_j)_{j\geq 0}$ in $\mathcal{M}$) is called a martingale (supermartingale, resp.) with respect to the filtration $(\mathcal{M}_j)_{0\leq j\leq n}$ if $a_j \in \mathcal{M}_j$ and $\mathcal{E}_j(a_{j+1}) = a_j$ ($\mathcal{E}_j(a_{j+1}) \leq a_j$, resp.) for every $j \geq 0$. Put $da_j = x_j - x_{j-1}$ ($j \geq 0$) with the convention that $a_{-1} = 0$. Then $da = (da_j)_{j\geq 0}$ is called the martingale difference of $(a_j)$. The reader is referred to [5] for more information.

Given a family of projections $(p_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{M}$, we denote by $\bigwedge_{\lambda\in\Lambda} p_\lambda$ the projection from $\mathcal{H}$ onto the closed subspace $\bigcap_{\lambda\in\Lambda} p_\lambda(\mathcal{H})$.

## 2 Main results

In the sequal, we assume that $(a_n)_{n\geq 0}$ is a self-adjoint martingale in $\mathcal{M}$ with respect to a filtration $(\mathcal{M}_n, \mathcal{E}_n)_{n\geq 0}$ with $x_0 = 0$ such that

$$d_n \leq 1 \quad \text{for all} \quad n \geq 1.$$

Put $b_0 = 0$ and $b_n = \sum_{k=1}^n \mathcal{E}_{k-1}(d_k^2)$, and for any positive number $t$, define

$$u_n^{(t)} := \exp\left\{ta_n - (e^t - 1 - t)b_n\right\}.$$

A generalization of the Lieb concavity [3], is proved by Araki [1] in the setting of von Neumann algebras as follows.

**Theorem 2.1** (Lieb–Araki concavity [1])**.** *Let $b \in \mathcal{M}$ be a self-adjoint operator. Then the function*

$$\phi : a \mapsto \tau\left(\exp(b + \log(a))\right)$$

*is concave on the strictly positive part of $\mathcal{M}$.*

**Remark 2.2.** The Jensen's inequality [4, Theorem A] states that if $\alpha$ is a unital positive map on $\mathcal{M}$ and $f$ is a real concave function on $[0, \infty)$, then for any self-adjoint element $a \in \mathcal{M}$ , it holds that $\tau\left(\alpha(f(a))\right) \leq \tau\left(f(\alpha(a))\right)$, where $f(a)$ is defined by the functional calculus. Now, let $b \in \mathcal{M}$ be self-adjoint and $a \in \mathcal{M}$ be a strictly positive operator. Define the continuous function $g : (0, \infty) \to (0, \infty)$ by $g(t) = \tau\left(\exp(b + \log(t + a))\right)$. It follows from the Lieb–Araki concavity that $g$ is concave. Let $\mathcal{E}_{\mathcal{N}}$ be any conditional expectation $\mathcal{E}_{\mathcal{N}}$ corresponding to a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$. Applying the Jensen's inequality, one may deduce that

$$\tau\left(\exp\left\{b + \log a\right\}\right) \leq \tau\left(\exp\left\{b + \log \mathcal{E}_{\mathcal{N}}(a)\right\}\right)$$

In the following result, we show that the sequence $(u_n^{(t)})_{n\geq 0}$ is trace-decreasing.

**Proposition 2.3.** *If the sequence $(a_n)_{n\geq 0}$, fixed as in the begining of section 2, is a positive martingale in $\mathcal{M}$, then $\tau(u_{n+1}^{(t)}) \leq \tau(u_n^{(t)})$ for all $n \geq 0$. Moreover, $\tau(u_n^{(t)}) \leq 1$ for all $n \geq 0$.*

Freedman's original proof and Tropp's approach are based on stopped martingales. However, an applicable version of noncommutative stopped martingales is not available. We present a noncommutative version of the Freedman inequality under some mild conditions as follows.

**Theorem 2.4.** *Let $\alpha \geq 0$ and $\beta > 0$ be real numbers and $(u_n^{(t_0)})_{n\geq 0}$ is a supermartingale in $\mathcal{M}$, where $t_0 = \log\left(\frac{\alpha+\beta}{\beta}\right)$. Then there is a sequence $(e_n)_{n\geq 1}$ of mutually orthogonal projections such that*

$$\sup_{n\geq 1} \frac{\tau\left(\mathbf{1}_{[\alpha,\infty)}(a_n) \wedge \mathbf{1}_{[0,\beta]}(b_n)\right)}{2^{n-1}} \leq \tau\left(\sum_{n=1}^{\infty} e_n\right) \leq \left(\frac{\beta}{\alpha+\beta}\right)^{\alpha+\beta} e^{\alpha}.$$

In what follows, we give an example [5] of the assumption that $u_n^{(t)}$ is a supermartingale. We use the software MATLAB for computations, not a proof.

**Example 2.5.** Let us consider the von Neumann algebra $\mathcal{M} = \mathbb{M}_2(\mathbb{C})$ of all $2 \times 2$ complex matrices with the identity $I_2$. Let $\tau := \frac{1}{2}\mathrm{tr}$ be the normalized trace on $\mathcal{M}$. Denote by $\mathcal{N}$ the subalgebra of diagonal matrices. Then

$$\mathcal{E}_{\mathcal{N}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Definer the filtration $(\mathcal{M}_n, \mathcal{E}_n)_{n\geq 1}$ by

$$\mathcal{M}_0 = \mathbb{C}I_2,\ \mathcal{E}_0(x) = \tau(x)I_2,\ \mathcal{M}_1 = \mathcal{N},\ \mathcal{E}_1 = \mathcal{E}_{\mathcal{N}},\ \text{and}\ \ \mathcal{M}_n = \mathcal{M},\ \mathcal{E}_n = \mathrm{id}_{\mathcal{M}}\ (n \geq 2).$$

If we set

$$a_0 := 0,\ a_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ a_2 := \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix},\quad \text{and} \quad a_n := a_2 \quad \text{for every } n \geq 2,$$

then clearly $(a_n)_{n\geq 0}$ is a self-adjoint martingale and $a_1 a_2 \neq a_2 a_1$. In addition,

$$d_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \leq 1,\ d_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \leq 1,\ \text{and}\ d_n = 0 \leq 1 \ \ (n \geq 3),$$

is the corresponding difference sequence. Moreover,

$$\begin{aligned}
b_0 &= 0, \\
b_1 &= \mathcal{E}_0(d_1^2) = \tau\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) I_2 = I_2 \\
b_2 &= \mathcal{E}_0(d_1^2) + \mathcal{E}_1(d_2^2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
b_n &= b_2 \ (n \geq 3).
\end{aligned}$$

Set $t = 2$ and $\lambda = e^2 - 3$, where $e$ is Euler's constant, we have

$$\begin{aligned}
u_0^{(2)} &= \exp\{2a_0 - \lambda b_0\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
u_1^{(2)} &= \exp\{2a_1 - \lambda b_1\} = \begin{pmatrix} e^{5-e^2} & 0 \\ 0 & e^{2-e^2} \end{pmatrix} \simeq \begin{pmatrix} 0.0917 & 0 \\ 0 & 0.0045 \end{pmatrix} \\
u_2^{(2)} &= \exp\{2a_2 - \lambda b_2\} = \exp\begin{pmatrix} 8 - 2e^2 & 2i \\ -2i & 4 - 2e^2 \end{pmatrix} \simeq \begin{pmatrix} 0.0022 & 0.0009i \\ -0.0009i & 0.0004 \end{pmatrix} \\
u_n^{(2)} &= u_2^{(2)} \quad (n \geq 3).
\end{aligned}$$

We have

$$\mathcal{E}_0(u_1^{(2)}) = 0.0962 I_2 \leq u_0^{(2)}$$

$$\text{and} \qquad \mathcal{E}_1\left(u_2^{(2)}\right) \simeq \begin{pmatrix} 0.0022 & 0 \\ 0 & 0.0004 \end{pmatrix} \leq \begin{pmatrix} 0.0917 & 0 \\ 0 & 0.0045 \end{pmatrix} = u_1^{(2)}.$$

Therefore, $(u_n^{(1)})_{n\geq 0}$ is a supermartingale.

One may conclude the classical Freedman inequality as follows.

**Corollary 2.6.** *Let $\{X_n : X_0 = 0, n \geq 0\}$ be a commutative martingale of bounded random variables. If the martingale difference sequence $\{D_n : D_0 = 0, n \geq 0\}$ satisfies $D_n \leq 1 \quad (n \geq 0)$, then for every $c \geq 0$ and $h > 0$, it holds that*

$$\text{Prob}\left(X_n \geq c \textit{ and } Y_n \leq h \text{ for some } n \geq 0\right) \leq \left(\frac{h}{c+h}\right)^{c+h} e^c, \tag{1}$$

*in which $Y_n = \sum_{k=1}^{n} \mathbb{E}_{k-1}(D_k^2)$.*

*Proof.* In the commutative case, the projection $\sum_{n=1}^{\infty} e_n = \bigvee_{n=1}^{\infty} e_n$, appeared in Theorem 2.4, is the indicator variable of

$$\begin{aligned} A &= \{\omega : X_n(\omega) \geq c \text{ and } Y_n(\omega) \leq h \text{ for some } n \geq 0\} \\ &= \bigcup_{n=1}^{\infty} \left( \{\omega : X_n(\omega) \geq c\} \cap \{\omega : Y_n(\omega) \leq h\} \right), \end{aligned}$$

where $e_n := \chi_{\{X_n \geq c\}} \chi_{\{Y_n \leq h\}}$. □

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