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n-Root of Matrices

R. Fallah-Moghaddam[1,*]
, [1]Department and Computer Science, University of Garmsar, Garmsar, Iran

Abstract. Assume that $A \in M_m(C)$, when C is an algebraically closed field. Also, consider that n is an arbitrary natural number. Here we are going to find a matrix $B \in M_m(C)$ with a construction method such that $B^n = A$.
Assume that $C \in M_m(\mathbb{H})$. Also, consider that t is an arbitrary natural number. We can find a matrix $D \in M_m(\mathbb{H})$ such that $D^t = C$.

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1. Introduction

In group theory, a multiplicative group G is called radicable if for every $a \in G$ and $n \in \mathbb{N}$, there exists an element $b \in G$ such that $b^n = a$. The adjective divisible is reserved for abelian G. . A division algebra D is called radicable if the unit group D^* is radicable. In particular, when D^* is abelian, the notion of divisibility and radicability coincide and D is called divisible (otherwise indivisible).

But, the structure of nonabelian radicable groups, in general, is unknown. The most extensive previous result in this direction was proved by Mahdavi-Hezavehi and Motiee. They determined in [3] and [4] the class of all radicable F-central division algebras D. In fact, for an indivisible field F, they showed that the following statements are equivalent:

(1) D is radicable;
(2) D contains a divisible subfield K/F;
(3) D is isomorphic to $(\frac{-1,-1}{F})$ (the ordinary quaternion division algebra) and $F(\sqrt{-1})$ is divisible.

In addition, G. ten Have (cf. [2]) gives a constructive way of finding roots in $M_m(K)$, where K is an arbitrary subfield of $\mathbb{C}$.

Given an algebraically closed field C, let $J_m(\lambda)$ denote the Jordan block of size $m \times m$ corresponding to eigenvalue $\lambda \in C$. When $\lambda \neq 0$ and the field C has characteristic zero, the Jordan form of $J_m(\lambda)^n$ is just also $J_m(\lambda^n)$. The above result holds when the field C

*Speaker. Email address: r.fallahmoghaddam@fmgarsar.ac.ir

has nonzero characteristic p and $m < p$. When $m \geqslant p$, things get messier. For instance, over $\mathbb{F}_2$, consider the case $m = 3$, when $n = 2$ and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

So,

$$A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, A^2 in this case has Jordan blocks of sizes smaller than 3.

Let K be a subfield of C. Then a matrix $A \in M_m(K)$ has a K-normal form $C_A = \bigoplus_{i=1}^{t} d_i C(f_i^{k_i})$, where the $f_i^{k_i}(x)$ are the elementary divisors in $K[x]$ of A. By the definition of elementary divisor the $f_i(x)$ are irreducible, and by using the aforementioned multiplicity d_i we can assume that no two of the $f_i^{k_i}(x)$ are the same. Note that, when m_i is the degree of $f_i(x)$, the order m of the matrix A is equal to $\sum_{i=1}^{t} d_i k_i m_i$. Now, for every i, take an $\alpha_i = {\alpha_i}^{(1)}$ which satisfies $f_i(\alpha_i) = 0$, and for $j = 1, \cdots, n$, take the n distinct values γ_{ij} such that $\gamma_{ij}^n = \alpha_i$. Let n_{ij} be given by $n_{ij} = [K(\gamma_{ij}) : K(\alpha_i)]$ for $i = 1, \cdots, t$ and $j = 1, \cdots, n$. In [1], it is proved that for a Given an algebraically closed field C, let K be a subfield of C, and let $A \in M_m(K)$ be nonsingular. Then, with notation as above,

(1) If $\mathrm{Char}(C) = 0$, then A has an n-th root in $M_m(K)$ if and only if there are nonnegative integers $b_{i1}, \ldots, b_{in}$, such that $d_i = b_{i1}n_{i1} + \cdots + b_{in}n_{in}$ for $i = 1, \ldots, t$.
(2) If $\mathrm{Char}(C) = p > 0$, $m < p$ and $p \nmid n$, then A has an n-th root in $M_m(K)$ if and only if there are nonnegative integers $b_{i1}, \ldots, b_{in}$, such that $d_i = b_{i1}n_{i1} + \cdots + b_{in}n_{in}$ for $i = 1, \ldots, t$.

In this article we will try to present a method to find the roots of a matrix in $M_m(\mathbb{C})$.

2. Main results

In the introduction, a brief overview of some of the steps taken to examine the root of a matrix is given. As we have seen, there are various theorems about the conditions under which the root of a matrix exists. In this section, we are going to present a constructive method for finding the $n-$roots of members of the ring $M_m(\mathbb{C})$. As mentioned in key theorem of Reference [1], every member of $M_m(\mathbb{C})$ has a $n-$roots, for any natural number n.

Main result (Constructive method). Assume that $A \in M_m(C)$, when C is an algebraically closed field. Also, consider that n is an arbitrary natural number. Here we are going to find a matrix $B \in M_m(C)$ with a construction method such that $B^n = A$.

We know that for any real number x, $(e^{\frac{1}{n} lnx})^n = x$. Now it suffices to generalize this concept to matrices.Since C is an algebraically closed field, then A has a Jordan canonical form. Without loss of genreality, it is enough to reduce the problem to Jordan form.

matrices. Therefore, we ay assume that:

$$A = \begin{bmatrix} a & 0 & . & . \\ 1 & a & 0 & . \\ . & . & . & . \\ . & 0 & 1 & a \end{bmatrix},$$

when a is an arbitrary element in C. Thus, $A = aI_m + N$, when $N \in M_m(C)$ such that $N^m = 0$. To simplify the calculations, we set $a = 1$, and thus Thus, $A = I_m + N$, when $N \in M_m(C)$ such that $N^m = 0$. Now, we define:

$$ln(A) = ln(I_m + N) = \sum_{i=1}^{m-1} \frac{(-1)^{i-1}}{i} N^i.$$

Now, set $B = (e^{\frac{1}{n} ln(A)})$. Then $B^n = A$, as we desired.

Assume that $\mathbb{H}$ is real quaternion division algebra over $\mathbb{R}$. By main result in [5], any $A \in M_m(\mathbb{H})$ has a canonical Jordan form in $M_{2m}(\mathbb{C})$. Of course, if we pay attention to the type of calculations used above and by the same argument as above, for any $A \in M_m(\mathbb{H})$ and $n \in \mathbb{N}$, we can find $B \in M_m(\mathbb{H})$ such that $B^n = A$.

3. Conclusion

Assume that $A \in M_m(\mathbb{H})$. Also, consider that n is an arbitrary natural number. We can find a matrix $B \in M_m(\mathbb{H})$ such that $B^n = A$.

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