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Improved weighted sum method for solving multiobjective quadratically constrained quadratic programming

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Abstract

Most of existing methods solving multiobjective quadratic programming with convex functions and linear constraints. In this paper, the improved weighted sum method is used to solve this problem with both convex and nonconvex quadratic functions. An algorithm is proposed, which converges to a set of efficient solutions.

Keywords: Multiobjective programming, Weighted sum method, Quadratic programming, Linear relaxation, Convex and concave envelop.

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1 Introduction

We consider the following problem:

$$\begin{array}{lll} \min f(x) = (f_1(x), ..., f_p(x)) & & \\ s.t. \qquad f_k(x) \geqslant 0, & k = p+1, ..., m, & \\ \qquad\quad x \in [a, b], & & \end{array} \tag{1}$$

where a, b are n dimensional vectors of nonnegative real numbers and $x \in [a, b]$ means that $a_i \leqslant x_i \leqslant b_i$ for all $i = 1, ..., n$. Each $f_k : \mathbb{R}^n \to \mathbb{R}$ is a quadratic function in the form of

$$f_k(x) = x^t H^k x + c_k^t x + d_k, \qquad k = 1, ..., p, \tag{2}$$

which $H^1, ..., H^p$ are real and symmetric $n \times n$ matrixes, $c_1, ..., c_p \in \mathbb{R}^n$ and $d_1, ..., d_p \in \mathbb{R}$.

Problem (1) is a multiobjective quadratically constrained quadratic programming (MQCQP) problem. If $p = 1$, it involves a single objective and we use the term SQCQP instead of MQCQP. In either case, the feasible set of problem (1) is denoted by $X := \{x \in \mathbb{R}^n | f_k(x) \geqq 0, k = p+1, ..., m, x \in [a, b]\}$ and the set $Y := \{y \in \mathbb{R}^p | y = f(x), x \in X\}$ called the image of X under f in the objective space.

MQCQP and SQCQP have been applied in many fields of science, including engineering, economics and etc (see, for example [3, 6]). At first we introduce some basic notations and definitions from [2, 5, 6]. Throughout the paper, $\mathbb{R}^n$ denotes the n dimensional Euclidean space. If $x, y \in \mathbb{R}^n$ then $x \leqq y (x < y)$ if and only if $x_i \leqslant y_i (x_i < y_i), \forall i = 1, ..., n$. In addition, $x \leq y$ means that $x \leqq y$ and $x \neq y$. We will denote by $\mathbb{R}^n_{\geqq}$ the set $\{x \in \mathbb{R}^n | x \geqq 0\}$.

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Definition 1.1. ([2]) Consider an MQCQP problem. The feasible solution $\hat{x} \in X$ is called efficient (weak efficient) if there is no another $x \in X$ such that $f(x) \leq f(\hat{x})(f(x) < f(\hat{x}))$. If $\hat{x} \in X$ is efficient (weak efficient) then $\hat{y} = f(\hat{x})$ is called a nondominated (weak nondominated) point. The set of all efficient solutions and nondominated points are called the efficient set and efficient frontier, respectively

The sets of weakly efficient solutions and efficient solutions are denoted by X_{wE} and X_E, respectively.

Definition 1.2. ([6]) A symmetric $n \times n$ matrix H is called

- Positive definite if and only if $x^t H x > 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$.
- Positive semidefinite if and only if $x^t H x \geqslant 0$ for all $x \in \mathbb{R}^n$.

Proposition 1.3. *([6]) Let C be a convex subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be twice continuously differentiable over $\mathbb{R}^n$.*

- *If $\bigtriangledown^2 f(x)$ (Hessian of f) is positive semidefinite for all $x \in C$, then f is convex over C.*
- *If $\bigtriangledown^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex over C.*

Corollary 1.4. *Consider the quadratic function $f(x) = x^t H x + c^t x + d$, where H is a symmetric $n \times n$ matrix, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Then, f is convex if the Hessian matrix H is positive semidefinite. Moreover, f is strictly convex if H is positive definite.*

Definition 1.5. ([6]) A function $h : \mathbb{R}^n_{\geqq} \longrightarrow \mathbb{R}$ is called an increasing function if $h(x) \leqslant h(y)$ for $x \leqq y$. It is a $d.m$ (difference of monotonic) function if $h(x) = h^+(x) - h^-(x)$, where h^+ and h^- are increasing functions.

Remark 1.6. Each quadratic function can be represented as a difference of two quadratic functions with nonnegative coefficients. So, every quadratic function is a $d.m$ function.

Definition 1.7. ([5]) Let X be a convex and compact subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$. The convex envelop of the function f over X is denoted by $Vex_X f$ and for all $x \in X$ is defined as

$$Vex_X f(x) = \sup\{g(x) : \text{g is convex on X},\ g(y) \leqslant f(y), \quad \forall y \in X\}$$

Definition 1.8. ([5]) Let X be a convex and compact subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$. The concave envelop of the function f over X is denoted by $Cav_X f$ and for all $x \in X$ is defined as

$$Cav_X f(x) = \inf\{g(x) : \text{g is convex on X},\ f(y) \leqslant g(y), \quad \forall y \in X\}$$

Theorem 1.9. *([1]) The convex envelop and concave envelop of the two dimensional bilinear function $f(x, y) = xy$ on the hyperrectangle $R = \{(x, y) \in \mathbb{R}^2 : \ell \leqslant x \leqslant u, m \leqslant y \leqslant M\}$ are respectively*

$$Vex_R(xy) = \max\{\ell y + mx - \ell m, uy + Mx - uM\},$$

$$Cav_R(xy) = \min\{\ell y + Mx - \ell M, uy + mx - um\}.$$

2 Improved weighted sum method

One of the well-known scalarization techniques in solving multiobjective optimization problems is the weighted sum method:

$$\min_{x\in X} \sum_{k=1}^{p} w_k f_k(x), \tag{3}$$

where $w = (w_1, w_2, \cdots, w_p) \in \mathbb{R}^p_{\geqq}$. For the scalarization model (3), the following results can be obtained.

Theorem 2.1. *([2]) If $\hat{x}$ is an optimal solution of (3) (and $w > 0$), then $\hat{x}$ is a weakly efficient (an efficient) solution of MQCQP (1).*

To get more results, the feasible set X is restricted by additional constraints such that each objective function is bounded from above. So, the improved weighted sum method is proposed ([4]):

$$\begin{array}{ll} \min\limits_{x\in X} & \sum\limits_{k=1}^{p} w_k f_k(x) \\ & f_k(x) \leqslant \epsilon_k, \qquad k = 1, ..., p, \end{array} \tag{4}$$

where $\epsilon = (\epsilon_1, ..., \epsilon_p)^t$ is an arbitrary vector in $\mathbb{R}^p$. The vector ϵ contains parameters $\epsilon_1, ..., \epsilon_p$, that can be determined by the decision-maker or an expert. Theoretically, these parameters can be any arbitrary value. Theorem 2.1 is still hold for any $\epsilon \in \mathbb{R}^p$. Also, we have

Theorem 2.2. *([4]) If $\hat{x}$ is an efficient (weakly efficient) solution of MQCQP (1). Then, there exist $w > 0$ $(w \geqq 0)$ and $\epsilon \in \mathbb{R}^p$, such that $\hat{x}$ is an optimal solution of (4).*

3 Main Results

Assume that $\hat{x}$ be a feasible solution of the MQCQP (1). Consider the scalarization problem (4) and set $\epsilon_k = f_k(\hat{x})$ for $k = 1, ..., p$. In this case, If $\hat{x}$ be an optimal solution of (4), then by Theorem 2.1, $\hat{x}$ is a (weakly) efficient solution of the problem (1). Therefore, to solve problem (1), at first, the scalarization problem (4) corresponding to the problem (1) is written as follows:

$$\begin{array}{lll} \min & \sum\limits_{k=1}^{p} w_k f_k(x) & \\ s.t. & f_k(\hat{x}) - f_k(x) \geqslant 0, & k = 1, ..., p, \\ & f_k(x) \geqslant 0, & k = p+1, ..., m, \\ & x \in [a, b], & \end{array} \tag{5}$$

An approach to find approximate solutions of the SQCQP (5) is to solve a linear relaxation of this problem. Here, we use a linear relaxation of problem (5), which is based on the convex and concave envelops of the bilinear terms in the quadratic functions $f_k(x)$. We denote this linear relaxation by $LP(a,b,\hat{x})$.

$$
\begin{array}{lll}
\min & \sum_{k=1}^{p} w_k f_k(x) & \\
s.t. & t_j^k \geqslant a_j H_j^k x + m_j^k x_j - a_j m_j^k, & j=1,...,n, \quad k=1,...,p, \\
& t_j^k \geqslant b_j H_j^k x + M_j^k x_j - b_j M_j^k, & j=1,...,n, \quad k=1,...,p, \\
& t_j^k \leqslant a_j H_j^k x + M_j^k x_j - a_j M_j^k, & j=1,...,n, \quad k=p+1,...,m, \\
& t_j^k \leqslant b_j H_j^k x + m_j^k x_j - b_j m_j^k, & j=1,...,n, \quad k=p+1,...,m, \\
& \sum_{j=1}^{n} t_j^k + c_k^t x + d_k \leqslant f_k(\hat{x}), & k=1,...,p, \\
& \sum_{j=1}^{n} t_j^k + c_k^t x + d_k \geqslant 0, & k=p+1,...,m, \\
& x \in [a,b], &
\end{array}
\tag{6}
$$

where t_j^k is the corresponding variable to the convex (concave) envelop of the bilinear function $x_j y_j^k$ such that $y_j^k = H_j^k x$ and H_j^k is the j-th row of the matrix H^k. Also m_j^k and M_j^k are the minimum and maximum of the linear function $H_j^k x$ on the interval $[a,b]$, respectively. Therefore,

$$
m_j^k = \min\{H_j^k x : x \in [a,b]\} = \sum_{q=1}^{n} \min\{H_{jq}^k a_q, H_{jq}^k b_q\},
$$

$$
M_j^k = \max\{H_j^k x : x \in [a,b]\} = \sum_{q=1}^{n} \max\{H_{jq}^k a_q, H_{jq}^k b_q\},
$$

wherein, H_{jq}^k is element (j,q) of the matrix H^k ([1]).
The next theorem shows that an optimal objective value of the linear programming problem (6) is a lower bound to an optimal objective value of the quadratic programming problem (5). At first, we prove the following lemma.

Lemma 3.1. *Assume that $\bar{x}$ be a feasible solution to the quadratic problem (5). Then there exists a vector $\bar{t} = (\bar{t}_1^1, ..., \bar{t}_n^m)$ such that $(\bar{x}, \bar{t})$ is a feasible solution of the linear problem (6).*

Theorem 3.2. *Assume $(x^\star, {t^\star}_1^1, ..., {t^\star}_n^m)$ be the optimal solution of the linear problem (6) and $\bar{x}$ be the optimal solution of the quadratic problem (5). Then*

(i) $\sum_{k=1}^{p} w_k f_k(x^\star) \leqslant \sum_{k=1}^{p} w_k f_k(\bar{x})$;

(ii) If $x^\star$ is a feasible solution for the quadratic problem (5), then
$\sum_{k=1}^{p} w_k f_k(x^\star) = \sum_{k=1}^{p} w_k f_k(\bar{x})$.

4 Proposed Algorithm

In the following, we propose an algorithm to solve problem (1) when $[a,b] \subseteq \mathbb{R}^2_{\geqq}$. At first, we divide the box $[a,b]$ into smaller sub boxes. Then, for each sub box, we solve the linear problem (6) to find a set of approximate (weakly) efficient solutions of the quadratic problem (1). By repeating this procedure and removing the non efficient solutions of this set at each iteration of the algorithm, we will have a better approximation of the efficient solutions set of problem (1).

Algorithm 4.1.

- **Input** $f = (f_1 = f_1^+ - f_1^-, ..., f_p = f_p^+ - f_p^-), w = (w_1, ..., w_p) \in \mathbb{R}^p_{\geq}, a = (a_1, a_2) \in \mathbb{R}^2_{\geqq}, b = (b_1, b_2) \in \mathbb{R}^2_{\geqq}$, positive integer m and positive real number Δ.

- **Initialization** $t := 1, [a^t, b^t] := [a, b], \mathcal{X}_E^{t-1} := \emptyset$.

- **Step 1** Divide rectangular $[a^t, b^t]$ into $(tm)^2$ subrectangular $[\ell_{ij}^t, u_{ij}^t]$ such that

$$\ell_{ij}^t = (a_1^t + (j-1)s_1^t,\ a_2^t + (i-1)s_2^t), \quad \text{and} \quad u_{ij}^t = (a_1^t + js_1^t,\ a_2^t + is_2^t),$$

for $i, j = 1, ..., tm$ and $s_r^t := \frac{b_r^t - a_r^t}{tm}$ for $r = 1, 2$.

- **Step 2** For each subrectangular $[\ell_{ij}^t, u_{ij}^t]$ for $i, j = 1, ..., tm$, solve the linear problem (6), where $[a, b] = [a^t, b^t]$ and $\hat{x}$ is an arbitrary point in $[\ell_{ij}^t, u_{ij}^t]$. Set $\mathcal{A} := \mathcal{A} \cup \{\bar{x}\}$, where $(\bar{x}, \bar{t}_1^1, ..., \bar{t}_n^m)$ is the optimal solution of (6).

- **Step 3** Construct the set $\mathcal{X}_E^t$ which is obtained by removing the non efficient points of problem (1) from $\mathcal{X}_E^{t-1} \cup \mathcal{A}$. Set, $\mathcal{X}_E := \mathcal{X}_E^t$.

- **Step 4** If $\frac{\|b^t - a^t\|}{tm} > \Delta$
  Set $t := t + 1, a^t := (a_1^t, a_2^t)$ and $b^t := (b_1^t, b_2^t)$ where $a_i^t = \min_{\bar{x} \in \mathcal{X}_E^t} \bar{x}_i$ and $b_i^t = \max_{\bar{x} \in \mathcal{X}_E^t} \bar{x}_i$, for $i = 1, 2$ then goto Step 1, else stop.
  end if.

- **Output** The sets $\mathcal{X}_E$ and $\mathcal{Y}_E := f(\mathcal{X}_E)$ as a discrete approximations of efficient set and efficient frointer set of problem (1), respectively.

In the sequel we prove that Algorithm 4.1 is convergent.

Theorem 4.2. *For each $\Delta > 0$, Algorithm 4.1 terminates after a finite number of iterations.*

The following example show the performance of the algorithm.

Example 4.3. Consider the following biobjective quadratic programming problem ([3]):

$$\begin{array}{ll} \min & (f_1(x), f_2(x)) \\ s.t. & -2x_1 - x_2 + 3 \leqslant 0, \\ & -x_1 - 2x_2 + 3 \leqslant 0, \\ & -2x_1 + 3x_2 - 3 \leqslant 0, \\ & x \in [(0.5, 0.5), (3, 3)], \end{array}$$

where $f_1(x) = 0.5(5x_1^2 + x_2^2)$ and $f_2(x) = 0.5(x_1^2 + 5x_2^2)$. By [3], the efficient set is two line segments between the points of $\{(\frac{3}{4}, \frac{3}{2}), (1, 1)\}$ and $\{(1, 1), (\frac{5}{3}, \frac{2}{3})\}$. Figure 1 shows the output of Algorithm 4.1 with $a = (0.5, 0.5), b = (3, 3), m = 7$ and $\Delta = 0.07$, in feasible space X and objective space Y.

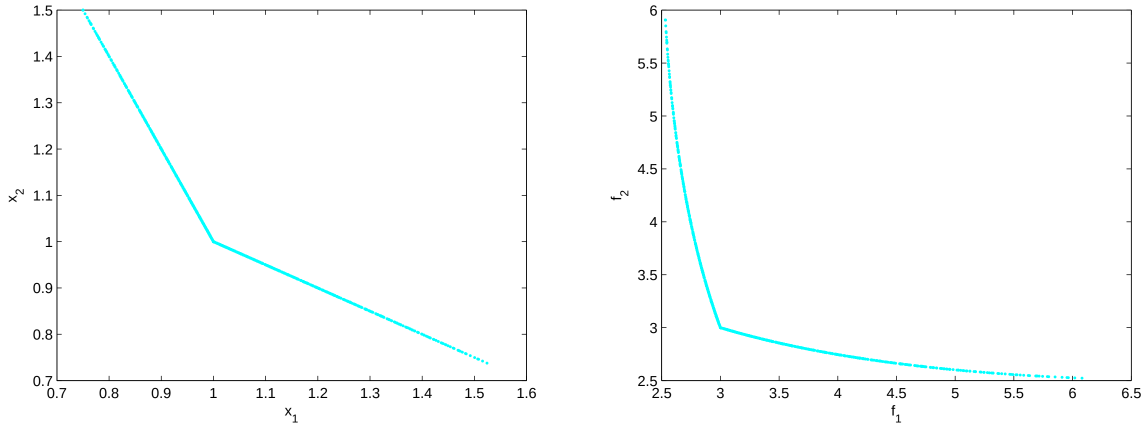

Figure 1: The sets $\mathcal{X}_E$ and $\mathcal{Y}_E$ for example 4.3.

5 Conclusion

While most of existing method for solving multiobjective quadratic problems consider convex objective functions and linear constraints, by the new version of the weighted sum method we solve this problem when the objective functions are either convex or nonconvex and constraints are in both linear and quadratic form. In fact, we convert problem (1) to an SQCQP by the improved weighted sum scalarization. A linear relaxation of SQCQP is extracted which calculate a lower bound for the optimal objective value of SQCQP on a given box. An algorithm is proposed to solve multiobjective quadratic problem with quadratic constraints when $[a, b] \subseteq \mathbb{R}^2_{\geqq}$. It terminates after a finite number of iterations.

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