The 11th International Seminar on
Linear Algebra and its Applications
Feb 2022

$K-$ uniformly Rotund Banach spaces

Payam Basiri[1]* and Zohreh normohamadi[2]

[1]Payame Noor University, Department of Mathemathics, Tehran, Iran.

[2]Bu -Ali Sina University, Department of Mathemathics, Hamedan , Iran.

Abstract

In this paper we obtain some of the basic geometric properties of $K-$UR spaces. Our purpose is to study the extent to which Banach space properties can be obtained by requiring a uniform behavior for all $n-$dimensional subspaces for some fixed $n \geq 2$.

Keywords: K-uniformly rotund space, Banach space, modulus of K-rotundity, volume of $k+1$ vectors.

Mathematics Subject Classification [2010]: 15A03, 15A15

1 Introduction

Let X be a real Banach space. According to von Neumanns famous geometrical characterizatin X is Hilbert space if and only if for all $x, y \in X$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Thus Hilbert space is distinguished among all real Banach spaces. For an arbitrary space X, one way of measuring the uniformly of the set of two dimensional subspaces is in terms of the real valued modulus of rotundity, i.e. for $\epsilon > 0$

$$\delta_X(\epsilon) = \inf\{2 - \|x+y\| \;:\; \|x\|, \|y\| \leq 1,\; \|x-y\| \geq \epsilon\}.$$

The space is said to be uniformly rotund if for each $\epsilon > 0$ we have $\delta_X(\epsilon) > 0$. Our purpose in this paper is to study the extent to which Banach space properties can be obtained by requiring a uniform behavior for all $n-$dimensional subspaces for some fixed $n \geq 2$. This idea is orginated with Milman [3] who discussed both smoothness and rotundity notation.

2 Main results

Now, we give some Basic definition which used in this paper [1]. Let X be a Banach space. For a nonempty subset A of X, The convex hull and the affine hull of A denote by $con(A)$ and $aff(A)$, respectively, and defined as follows

$$con(A) := \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \geq 0, \quad for \quad i = 1, 2, ..., n \quad \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}\},$$

*Speaker. Email address: payambaissiri@pnu.ac.ir

$$aff(A) := \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \in \mathbb{R}, \quad for \quad i = 1, 2, ..., n \quad \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}\}.$$

The set A is called affine if $aff(A) = A$. One can easily show that every affine set corresponds with a subspace.
Let A be a affine set and Y be its corresponding subspace. Then the dimension of A is defined the same dimension of Y. Further, the dimension of a convex set A is defined as the dimension of the smallest affine set which contains A.
Given $x_1, x_2, ..., x_k \in X$, Sliverman [5], introduced the concept of volume of $k+1$ vectors, denote by $V(x_1, x_2, ..., x_{k+1})$ and defined by

$$V(x_1, x_2, ..., x_{k+1}) = \frac{1}{k!} \sup \left\{ \begin{vmatrix} f_1(x_2 - x_1) & \dots & f_1(x_{k+1} - x_1) \\ f_2(x_2 - x_1) & \dots & f_2(x_{k+1} - x_1) \\ \vdots & \vdots & \vdots \\ f_k(x_2 - x_1) & \dots & f_k(x_{k+1} - x_1) \end{vmatrix} : f_1, ..., f_k \in B_{X^*} \right\}.$$

By one of the consequence of Han-Banach theorem, we get $V(x_1, x_2) = \|x_1 - x_2\|$ for any $x_1, x_2 \in X$.
Note that $V(x_1, x_2, ..., x_{k+1}) = 0$ iff the dimension of the convex hull of $\{x_1, x_2, ..., x_{k+1}\}$ dose not exceed $k - 1$.
Using the notion of volume of $k + 1$ vectors, Sullivan [6] defined the concept of $K-$uniform rotund spaces.

Definition 2.1. [4] Let X be a Banach space and $k \in \mathbb{N}$. Set

$$\mu_X^{(k)} = \sup\{V(x_1, ..., x_{k+1}) : x_1, ..., x_{k+1} \in B_X, i = 1, 2, ..., k + 1\}.$$

The function $\delta_X^{(k)} : [0, \mu_X^{(k)}) \longrightarrow [0, 1]$ is said to the modulus of K-rotundity of X and define by

$$\delta_X^{(k)}(\varepsilon) = \inf \left\{ 1 - \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i\| : x_1, ..., x_{k+1} \in B_X \quad \text{and} \quad V(x_1, ..., x_{k+1}) \geq \varepsilon \right\},$$

where $\epsilon \in [0, \mu_X^{(k)})$.

The Banach space X is said to be K-uniformly rotund (in short $K - UR$) if $\delta_X^{(k)} > 0$ for any $\epsilon \in (0, \mu_X^{(k)})$.
Note that a Banach space X is said to be $1-$ UR if, for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if x and y are norm-1 vector with $\|x + y\| \geq 2 - \delta(\epsilon)$ then

$$\sup \left\{ \begin{vmatrix} 1 & 1 \\ f(x) & f(y) \end{vmatrix} : f \in B_{X^*} \right\} < \epsilon.$$

By analogy we say that X is $2-$ UR if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for all norm-1 $x, y, z \in X$, if $\|x + y + z\| > 3 - \delta(\epsilon)$ then

$$\sup \left\{ \begin{vmatrix} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ g(x) & g(y) & g(z) \end{vmatrix} : f, g \in B_{X^*} \right\} < \epsilon.$$

Notice that of $2-$UR the quantity in the brackets can be thought of as twice the area of the triangle with vertices at x, y and z. In geometrical terms, if three points on the surface of $2-$UC space enclose an area $\geq \frac{\epsilon}{2}$ then the centroid of the triangle they determine lies a distance at least $(\frac{\epsilon}{3})$ beneath the surface of the ball. [5]

Theorem 2.2. *[6] if for some k a Banach space X is $K-UR$ then X is $K+1-UR$.*

Proof. Suppose that there are norm-1 sequences $(x_n^{(1)}), (x_n^{(2)}), ..., (x_n^{(k+2)})$ with $\|x_n^{(1)} + x_n^{(2)} + ... + x_n^{(k+2)}\| \to k+2$. Then from the triangle inequality for each j we have

$$\|x_n^{(1)} + ... + x_n^{(j-1)} + x_n^{(j+1)} + ... + x_n^{(k+2)}\| \to k+1.$$

Now, let $f_1, f_2, ..., f_{k+1} \in \mathcal{B}_{X^*}$ and consider the determinant

$$\begin{vmatrix} 1 & 1 & & 1 \\ f_1(x_n^{(1)}) & f_1(x_n^{(2)}) & \dots & f_1(x_n^{(k+2)}) \\ \vdots & \vdots & & \vdots \\ f_{k+1}(x_n^{(1)}) & f_{k+1}(x_n^{(2)}) & \dots & f_{k+1}(x_n^{(k+2)}) \end{vmatrix}$$

expanding in minors along the second row and the fact that X is $K-$UR we conclude that X is $K+1-$UR . ☐

The converse of above theorem is not true. For example, the Banach space $l^{p,1}(\mathbb{N})$ for $1 < p < \infty$ is $2-$UR but not $1-$UR where $l^{p,1}(\mathbb{N})$ is the $l^p(\mathbb{N})$ space with suitable renorm.

Corollary 2.3. *[2] Let X be a Banach space. Then $\delta_X^{(k)}(\epsilon)$ is continuous on $[0, \mu_X^{(k)})$*

Lemma 2.4. *[4]. Let X be a K-uniformly rotund Banach space for some $k \in \mathbb{N}$. and $x_1, ..., x_{k+1} \in B_X$ such that $V(x_1, ..., x_{k+1}) = \epsilon > 0$ Then*

$$\|t_1x_1 + t_2x_2 + ... + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, ..., t_{k+1}\}\delta_X^k(\varepsilon),$$

where $\sum_{i=1}^{k+1} t_i = 1$, $t_i \geq 0$ for $i = 1, 2, ..., k+1$.

Proof. Without loss of generality, we can assume that $t_1 = \min\{t_1, t_2, ..., t_{k+1}\}$.

$$\begin{aligned}
&\|t_1x_1 + t_2x_2 + ... + t_{k+1}x_{k+1}\| \\
&= \|t_1(x_1 + ... + x_{k+1}) + (t_2 - t_1)x_2 + (t_3 - t_1)x_3 + ... + (t_{k+1} - t_1)x_{k+1}\| \\
&\leq (k+1)t_1\|\frac{x_1 + x_2 + ... + x_{k+1}}{k+1}\| + (t_2 - t_1)\|x_2\| + (t_3 - t_1)\|x_3\| + ... + (t_{k+1} - t_1)\|x_{k+1}\| \\
&\leq (k+1)t_1(1 - \delta_X^{(k)}(\epsilon) + t_2 + t_3 + ... + t_{k+1} - kt_1 \\
&= (k+1)t_1 - (k+1)t_1\delta_X^{(k)}(\epsilon) + 1 - (k+1)t_1 \\
&= 1 - (k+1)t_1\delta_X^{(k)}(\epsilon)
\end{aligned}$$

Hence $\|t_1x_1 + t_2x_2 + ... + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, ..., t_{k+1}\}\delta_X^k(\varepsilon)$. ☐

3 Conclusion

Motivated by definition of $K-$UR space we can extend application of uniformly properties of Banach spaces. For example the existence of fixed point problem in the setting of uniformly rotund Banach space extended by Radhakrishnan in [4].

References

[1] R. P. Agarwal, D.O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitizian-type Mappings with Applications*, Springer, new York, 2009.

[2] T. C. Lim, *On moduli of k-convexity*, Abstr. Appl. Anal. **4** (1999), 243-247.

[3] D. P. Milman and M. S. Brodskii, *On the center of a convex set*, Dokl. Akad. Nauk. SSR(N.S) 59(1948), 837-840.

[4] M. Radhakrishnan, S. Rajesh and Sushama Agrawal, *Some fixed point theorems on non-convex sets*, Appl. Gen. Topol. **18**, no. 2 (2017), 377-390.

[5] E. Sliverman, *Definitions of Lebesgue area for surfaces in metric spaces*, RivistaMat. Univ. Parma **2** (1951), 47-76.

[6] F. Sullivan, *A generalization of uniformly rotund Banach spaces*, Canada. J. Math. **31** (1979), 628-636.