The 11th International Seminar on Linear Algebra and its Applications
Feb 2022

A note on generalized Jordan derivations of triangular rings

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Abstract

In this talk first of all we reveiw some results concerning (generalized) Jordan derivations on triangular ring. Then under some conditions we shall prove that every generalized Jordan derivation associate with Hochschild 2-cocycle α on a triangular ring is a generalized derivation associate with Hochschild 2-cocycle α.

Keywords: generalized derivation, generalized Jordan derivation, Hochschild 2-cocycle, Triangular ring

Mathematics Subject Classification [2010]: 16W25

1 Introduction

Let $\mathcal{R}$ be a ring, an additive mapping $d : \mathcal{R} \longrightarrow \mathcal{R}$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in \mathcal{R}$. Obviously, every derivation is a Jordan derivation. But the inverse is in general not true. The standard problem to find out whether (under some conditions) a Jordan derivation is necessarily a derivation. The first result in this direction was due to Herstein. In 1957, he proved that every Jordan derivation on a 2-torsion free prime ring is a derivation [3]. In 1988, Brešar generalized Herstein's result to Jordan derivations of semprime rings [1].

An additive mapping $f : \mathcal{R} \longrightarrow \mathcal{R}$ is called a generalized derivation(resp. generalized Jordan derivation) if there exists a derivation $d : \mathcal{R} \longrightarrow \mathcal{R}$ such that $f(xy) = f(x)y + xd(y)$, (resp. $f(x^2) = f(x)x + xd(x)$)
for all $x, y \in \mathcal{R}$. We denote it by (f, d). (see Brešar [2])

Ispired by Brešar idea, several outhers have introduced various copies of "generalized" derivations. Recently, Nakajima [4] has introduced a new type of generalized derivations and generalized Jordan derivations associated with Hochschild 2-cocycle in the following way. Let $\mathcal{R}$ be a ring and $\mathcal{M}$ be an $\mathcal{R}$-bimodule and x, y, z be arbitrary elements of $\mathcal{R}$. Let $\alpha : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{M}$ be a biadditive map, that is, an additive map on each components α is called a **Hochschild 2-cocycle** if

$$x\alpha(y, z) - \alpha(xy, z) + \alpha(x, yz) - \alpha(x, y)z = 0$$

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An additive map $f : \mathcal{R} \longrightarrow \mathcal{M}$ is called a generalized derivation if there exists a 2-cocycle α such that

$$f(xy) = f(x)y + xf(y) + \alpha(x, y)$$

and f is called a generalized Jordan derivation if

$$f(x^2) = f(x)x + xf(x) + \alpha(x, x)$$

Examples

(1) Let $d : \mathcal{R} \longrightarrow \mathcal{M}$ be a derivation, if (f, d) is a generalized derivation, then the map $\alpha_1 : \mathcal{R} \times \mathcal{R} \ni (x, y) \mapsto x(d - f)(y) \in \mathcal{M}$ is biadditive and satisfy the 2-cocycle condition. Then f is a generalized derivation associated to α_1.

(2) If $f : \mathcal{R} \longrightarrow \mathcal{M}$ is left multiplier, that is, f is additive and $f(xy) = f(x)y$, then by $f(xy) = f(x)y + xf(y) + x(-f)(y)$, we have 2-cocycle $\alpha_2 : \mathcal{R} \times \mathcal{R} \ni (x, y) \mapsto x(-f)(y) \in \mathcal{M}$ and f is a generalized derivation associated to α_2.

(3) Let f be a (σ, τ)- derivation, that is, σ and τ is a ring homomorphisms of $\mathcal{R}$ and $f(xy) = f(x)\sigma(y) + \tau(x)f(y)$. Then the map $\alpha_3 : \mathcal{R} \times \mathcal{R} \ni (x, y) \mapsto f(x)(\sigma(y) - y) + (\tau(x) - x)f(y) \in \mathcal{M}$ is biadditive and satisfies the 2-cocycle condition. Since

$$f(xy) = f(x)y + xf(y) + \alpha_3(x, y),$$

(σ, τ)- derivation f is a generalized derivation associated to α_3 .

Lemma 1.1. $[4, Lemma2]$. *Let $(f, \alpha) : \mathcal{R} \longrightarrow \mathcal{M}$ be a generalized Jordan derivation associate with Hochschild 2-cocycle α and $\mathcal{M}$ a 2-torsion free module.Then the following relations hold:*

$$\begin{aligned}
&(1)f(xy + yx) = f(x)y + xf(y) + \alpha(x, y) + f(y)x + yf(x) + \alpha(y, x);\\
&(2)f(xyx) = f(x)yx + xf(y)x + xyf(x) + x\alpha(y, x) + \alpha(x, yx);\\
&(3)f(xyz + zyx) = f(x)yz + xf(y)z + xyf(z) + x\alpha(y, z) + \alpha(x, yz)\\
&+f(z)yx + zf(y)x + zyf(x) + z\alpha(y, x) + \alpha(z, yx).
\end{aligned}$$

In 2006, Zhang and Yu [5] showed that every Jordan derivation on a triangular algebra is a derivation. More precisely, they proved the following result.

Theorem 1.2. [5] *Let $\mathcal{A}, \mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$ and let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule that is faitful as a left $\mathcal{A}$-module and as a right $\mathcal{B}$-module. Then every Jordan derivation on the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a derivation.*

Recall that a triangular ring $\mathfrak{T} = Tir(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a ring of the form

$$\mathfrak{T} = Tir(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}; a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

We set

$$\mathfrak{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}; a \in \mathcal{A} \right\}$$

,

$$\mathfrak{T}_{12} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}; m \in \mathcal{M} \right\}$$

and

$$\mathfrak{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}; b \in \mathcal{B} \right\}$$

Then we may write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$ and every elemet $\mathcal{A} \in \mathfrak{T}$ can be writen as $\mathcal{A} = A_{11} + A_{12} + A_{22}$.Here, $A_{ij} \in \mathfrak{T}_{ij}, i, j \in \{1, 2\}$.

In this talk under some conditions we shall prove that every generalized Jordan derivation associate with Hochschild 2-cocycle α on a triangular ring is a generalized derivation associate with Hochschild 2-cocycle α.

2 Main results

The following theorem is the main purpose of this paper.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free semiprime rings and let $\mathcal{M}$ be a faithful non-degenerate $(\mathcal{A}, \mathcal{B})$-bimoudle then every generalized Jordan derivation f associate with Hochschild 2-cocycle α on triangular ring $\mathfrak{T} = Tir(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a generalized derivation associate with Hochschild 2-cocycle α .*

We recall that $(\mathcal{A}, \mathcal{B})$-module $\mathcal{M}$ is called non-degenerate if for all $x \in \mathcal{M}$ $\mathcal{A}x = 0$ implies $x = 0$ and $x\mathcal{B} = 0$ implies $x = 0$.

For the proof of the above theorem we need the following lemmas.

Lemma 2.2. *Let $\alpha : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ be a Hochschild 2-cocycle. Then for any $A_{11}, B_{11} \in \mathfrak{T}_{11}$, $A_{12}, B_{12}, C_{12} \in \mathfrak{T}_{12}$ and $A_{22}, B_{22} \in \mathfrak{T}_{22}$, we have*

1) $B_{22}\alpha(A_{11}, B_{22}) = \alpha(B_{22}, A_{11})B_{22}$,

2) $B_{11}\alpha(B_{22}, A_{11}) = \alpha(B_{11}, B_{22})A_{11}$,

3) $B_{22}\alpha(B_{12}, A_{11}) = \alpha(B_{22}, B_{12})A_{11}$,

4) $B_{22}\alpha(A_{12}, B_{22}) + \alpha(B_{22}, A_{12}B_{22}) = \alpha(B_{22}, A_{12})B_{22}$,

5) $A_{11}\alpha(B_{22}, A_{12}) = \alpha(A_{11}, B_{22})A_{12}$,

6) $A_{12}\alpha(B_{12}, B_{22}) + \alpha(A_{12}, B_{12}B_{22}) = \alpha(A_{12}, B_{12})B_{22}$,

7) $A_{12}\alpha(B_{12}, A_{11}) = \alpha(A_{12}, B_{12})A_{11}$,

8) $B_{22}\alpha(A_{12}, B_{12}) = \alpha(B_{22}, A_{12})B_{12}$,

9) $A_{11}\alpha(B_{11}, C_{12}) - \alpha(A_{11}B_{11}, C_{12}) + \alpha(A_{11}, B_{11}C_{12}) = \alpha(A_{11}, B_{11})C_{12}$,

10) $C_{12}\alpha(A_{22}, B_{22}) = \alpha(C_{12}A_{22}, B_{22}) - \alpha(C_{12}, A_{22}B_{22}) + \alpha(C_{12}, A_{22})B_{22}$.

Lemma 2.3. *For arbitrary $A_{11} \in \mathfrak{T}_{11}$ and $B_{22} \in \mathfrak{T}_{22}$, we have:*

$$\begin{aligned}
(i) \qquad & [f(B_{22})]_{11}A_{11} = -[\alpha(B_{22}, A_{11})]_{11}, \\
& B_{22}[f(A_{11})]_{22} = -[\alpha(B_{22}, A_{11})]_{22}, \\
& [\alpha(B_{22}, A_{11})]_{12} = 0. \\
(ii) \qquad & f(B_{22}A_{11}) = f(B_{22})A_{11} + B_{22}f(A_{11}) + \alpha(B_{22}, A_{11}) \\
(iii) \qquad & f(A_{11}B_{22}) = f(A_{11})B_{22} + A_{11}f(B_{22}) + \alpha(A_{11}, B_{22})
\end{aligned}$$

Lemma 2.4. *For any $B_{12} \in \mathfrak{T}_{12}$, the following is ture:($A_{11} \in \mathfrak{T}_{11}$)*

$$\begin{aligned}
& [f(B_{12})]_{11}A_{11} = -[\alpha(B_{12}, A_{11})]_{11}, \\
& [\alpha(B_{12}, A_{11})]_{22} = 0, \\
& B_{12}[f(A_{11})]_{22} = -[\alpha(B_{12}, A_{11})]_{12}.
\end{aligned}$$

Lemma 2.5. *For any $A_{11} \in \mathfrak{T}_{11}$ and $B_{12} \in \mathfrak{T}_{12}$, we have*

$$\begin{aligned}
& (i) f(B_{12}A_{11}) = f(B_{12})A_{11} + B_{12}f(A_{11}) + \alpha(B_{12}, A_{11}), \\
& (ii) f(A_{11}B_{12}) = f(A_{11})B_{12} + A_{11}f(B_{12}) + \alpha(A_{11}, B_{12}).
\end{aligned}$$

Lemma 2.6. *For any* $A_{12} \in \mathfrak{T}_{12}$ *, the following is ture:*$(B_{22} \in \mathfrak{T}_{22})$

$$\begin{aligned}
&[f(B_{22})]_{11}A_{12} = -[\alpha(B_{22}, A_{12})]_{12}, \\
&[\alpha(B_{22}, A_{12})]_{11} = 0, \\
&B_{22}[f(A_{12})]_{22} = -[\alpha(B_{22}, A_{12})]_{22}.
\end{aligned}$$

Lemma 2.7. *For any* $A_{12} \in \mathfrak{B}_{12}$ *and* $B_{22} \in \mathfrak{B}_{22}$*, we have*

$$\begin{aligned}
&(i) f(B_{22}A_{12}) = f(B_{22})A_{12} + B_{22}f(A_{12}) + \alpha(B_{22}, A_{12}), \\
&(ii) f(A_{12}B_{22}) = f(A_{12})B_{22} + A_{12}f(B_{22}) + \alpha(A_{12}, B_{22}).
\end{aligned}$$

Lemma 2.8. *For any* $A_{12}, B_{12} \in \mathfrak{T}_{12}$ *, the following is ture:*

$$\begin{aligned}
&[f(A_{12})]_{11}B_{12} + A_{12}[f(B_{12})]_{22} = -[\alpha(A_{12}, B_{12})]_{12}, \\
&[\alpha(A_{12}, B_{12})]_{11} = 0, \\
&[\alpha(A_{12}, B_{12})]_{22} = 0.
\end{aligned}$$

Lemma 2.9. f *is a generalized derivation on* $\mathfrak{T}_{12}$*.*

Lemma 2.10. f *is a generalized derivation on* $\mathfrak{T}_{11}$ *and* $\mathfrak{T}_{22}$*.*

Applying the above lemmas, we have for any $A = A_{11} + A_{12} + A_{22}$ and $B = B_{11} + B_{12} + B_{22}$ in $\mathfrak{T}$.

$$\begin{aligned}
f(AB) &= f(\sum_{1=i\leq j=2} A_{ij} \sum_{1=k\leq l=2} B_{kl}) \\
&= f(\sum_{\substack{1=i\leq j=2 \\ 1=k\leq l=2}} A_{ij}B_{kl}) \\
&= \sum_{\substack{1=i\leq j=2 \\ 1=k\leq l=2}} f(A_{ij}B_{kl}) \\
&= \sum_{\substack{1=i\leq j=2 \\ 1=k\leq l=2}} (f(A_{ij})B_{kl} + A_{ij}f(B_{kl}) + \alpha(A_{ij}, B_{kl})) \\
&= \sum_{1=i\leq j=2} f(A_{ij})B + A \sum_{1=k\leq l=2} f(B_{kl}) + \sum_{\substack{1=i\leq j=2 \\ 1=k\leq l=2}} \alpha(A_{ij}, B_{kl}) \\
&= f(\sum_{1=i\leq j=2} A_{ij})B + Af(\sum_{1=k\leq l=2} B_{kl}) + \sum_{\substack{1=i\leq j=2 \\ 1=k\leq l=2}} \alpha(A_{ij}, B_{kl}) \\
&= f(A)B + Af(B) + \alpha(A, B).
\end{aligned}$$

The proof of the Theorem is complete.

3 Conclusion

We extended some results concerning to (generalized) Jordan derivations, to generalized Jordan derivations associated to a Hochschild 2-cocycle.

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