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Decomposability of multivariate majorization

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Abstract

Let $x, y \in \mathbb{R}^n$. We use the notation $x \prec y$ when x is multivariate majorized by y. We say that $x \prec y$ is decomposable at k $(1 \leq k < n)$ if $x \prec y$ has a coincidence at k and $y_k \neq y_{k+1}$. Corresponding to this majorization we have a doubly stochastic matrix D such that $x = Dy$. Levow proved that if $x \prec y$ is decomposable at some k $(1 \leq k < n)$ then D is of the form $D_1 \oplus D_2$ where D_1 and D_2 are doubly stochastic matrices, this paper presents the converse of this theorem.

Keywords: Decomposable, Doubly stochastic matrix, Majorization

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1 Introduction

Let $\mathbf{M}_n$ be the set of all real matrices of order n. A matrix $D \in \mathbf{M}_n$ of nonnegative real numbers for which the sums of the entries in each row and each column are all one is said to be *doubly stochastic*. We denote the set of all doubly stochastic matrices of order n by Ω_n.
Let $\mathbb{R}^n$ be the set of all $n-$tuples $(x_1, \ldots, x_n)$ of real numbers. Also, let $x^\downarrow$ be the vectors obtained by rearranging the coordinates of $x \in \mathbb{R}^n$ in the decreasing order. Thus, if $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)$, then $x_1^\downarrow \geq \cdots \geq x_n^\downarrow$.
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We say that x is *majorized* (*multivariate majorized*) by y, in symbols $x \prec y$, if

$$\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow, \qquad 1 \leq k \leq n,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

For further information about majorization, we refer the reader to [1–5].

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2 Main results

The following concepts are defined by Levow in [4].
Let $x, y \in \mathbb{R}^n$, $x \prec y$, and

$$\delta_k = \sum_{i=1}^{k} (y_i^{\downarrow} - x_i^{\downarrow}), 1 \leq k \leq n-1, \tag{1}$$

then $\delta_k \geq 0$.
If $\delta_k = 0$, we say that $x \prec y$ has a *coincidence* at k. If $x \prec y$ has a coincidence at k and $y_k \neq y_{k+1}$, we say that $x \prec y$ is *decomposable* at k.

Theorem 2.1 ([5, A.4]). *Let $x, y \in \mathbb{R}^n$, then $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix $D \in \boldsymbol{M}_n$*

If $x \prec y$ and $y = (D_1 \oplus D_2)x$, where $D_1 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$, then it is clear from the theorem of [2] that $x \prec y$ has a coincidence at k. Also, if $x \prec y$ has a coincidence at k, it is clear from the Theorem2.1 that $D = D_1 \oplus D_2$ where $D_1 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$ such that $x = Dy$. In the Theorem2.2 Levow proves that if $x = Dy$, the decomposability of $x \prec y$ at k guarantees that D is a direct sum of matrices in Ω_k and Ω_{n-k}.

Theorem 2.2 ([4, Theorem 2]). *Suppose that $x \prec y$ is decomposable at k and that $y = Dx$ then there are matrices $D_1 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$ such that $D = D_1 \oplus D_2$.*

In [4], Levow proved that decomposability of $x \prec y$ implies that if $x = Dy, D \in \Omega_n$, then D is the direct sum of two doubly stochastic matrices. Here, we study majorization and related doubly stochastic matrices. We also present that decomposability of $x \prec y$ is a necessary condition for D to be a direct sum of $D_1 \oplus D_2$, where $x = Dy$ and $D_1, D_2 \in \Omega_n$.

Theorem 2.3. *Let $x, y \in \mathbb{R}_+^n$ with $x \prec y$. If there exists some k $(1 \leq k \leq n)$ that for every $D \in \boldsymbol{M}_n$ such that $x = Dy$, we have $D = D_1 \oplus D_2$ where $D_1, D_2 \in \Omega_k$ and $D_2 \in \Omega_{n-k}$, then $x \prec y$ is decomposable at k.*

Proof. Let $x \prec y$. Then $x = Dy$ for some doubly stochastic matrix D. The hypothesis ensures that $D = D_1 \oplus D_2$ where $D \in \Omega_k$ and $D_2, \in \Omega_{n-k}$.

The relation $x = Dy$ ensures that $(x_1, \ldots, x_k) = D_1(y_1, \ldots, y_k)$ where $D_1 \in \Omega_k$, and so $(x_1, \ldots, x_k) \prec (y_1, \ldots, y_k)$. It follows that $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i$. Therefore, $x \prec y$ has a coincidence at k.
Now, we claim that $y_k \neq y_{k+1}$. If not; $y_k = y_{k+1}$. We will constract a matrix D' such that $x = D'y$, but D' is not as a direct sum of two doubly stochastic matrices.
Set $D = [D^1 D^2 \ldots D^n]$, where D^i is the ith column of the matrix D. Now, define $D' = [D^1 \ldots D^{k-1} D^{k+1} D^k D^{k+2} \ldots D^n]$. We observe that $x = D'y$. As $y_k = y_{k+1}$ and D has the form given in the hypothesis, we see that D' has the same form which we wanted to create. It is a contradiction. Hence $y_k \neq y_{k+1}$, and so $x \prec y$ is decomposable at k. □

3 Conclusion

Decomposability is some conditions on the majorization relation $\prec$ on vectors. Here we prove the converse of a theorem of Levow. Examining the decomposability condition on other majorization relation can be subject of future research works.

References

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