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Extremal transmission irregular trees with respect to Wiener index

Yaser Alizadeh[1] and Zohre Molaee[2*]

[1,2] Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran

Abstract

Let G be a graph. G is called transmission irregular (shortly TI graph) graph if no two row sum are the same in the distance matrix of G. A family of TI trees with three branches are presented and extremal TI trees with respect to the Wiener index are determined.

Keywords: Distance matrix, Wiener index, transmission irregular graph;

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1 Introduction

throughout the paper we consider only simple connected graphs. Let $G(V,E)$ be a graph of order n with vertex set $V(G) = \{v_1, v_2 \cdots v_n\}$. Degree of vertex v, $deg(v)$ is the number of vertices adjacent to v. Adjacency matrix of G is a square matrix of order n whose ij-th entry A_{ij} is 1 if vertices v_i and v_j are adjacent else $A_{ij} = 0$. A k-regular graph is a graph whose all vertices have same degree k. In the other words sum of entries of each column or row equals k. Let $\Delta(G)$ denotes a diagonal matrix whose i-th entry is $deg(v_i)$. Laplacian matrix of G, $L(G)$ definesd as $L(G) = \Delta(G) - A(G)$. Let $\lambda_1 \geq \lambda_2 \cdots \lambda_n$ be eigenvalues of $L(G)$. It is well known fact that $\lambda_n = 0$ for all graphs and for connected graphs $\lambda_{n-1} > 0$ that is named algebraic connectivity. The theory of Laplacian spectra of graphs has been extensively studied. Readers reffered to [9,11] for a review. Let v and w be two vertices of G. distance $d(u,v)$ is the length of shortest path connecting u to v. Distance matrix of G, $D(G)$ is a square matrix of order n with $D_{ij} = d(v_i, v_j)$. Transmission of v, $Tr_G(v)$ is sum of distances between v and other vertices of G. Abviously sum of entries $i-$th row or column of distance matrix equals transmission of v_i. Let Γ be set of all graphs. A topological index, I is function from Γ to real numbers such that if G and H are two isomorphic graph then $I(G) = I(H)$. A well known topological index based on distance in graph is Wiener index introduced as sum of all distances between pair vertices of a graph.

$$W(G) = \sum_{\{u,v\} \subset V(G)} d(u,v).$$

The Wiener index can be presented as

$$W(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{ij} = \frac{1}{2} \sum_{v \in V(G)} Tr(v)$$

*Speaker. Email address: zohremolaeegm@gmail.com

An interesting result with respect to Wiener index and Laplacian spectrum on trees was communicated independently in [8, 10–12] as follows:

Theorem 1.1. *Let T be a tree of order n with Laplacian spectrum $\lambda_1 \geq \lambda_2 \cdots \lambda_n$ then*

$$W(T) = \frac{1}{n}\sum_{i=1}^{n-1}\frac{1}{\lambda_i}$$

Wiener complexity of G, $C_w(G)$ is the number of different vertex transmissions of G. A graph with $C_w(G) = 1$ is called transmission regular graph. Also a graph is called tarnsmission irregular graph if each two vertices have different transmission. In the other words $C_w(G) = n$ where n denotes the order of G. It is well knwon fact proved by Moor and Moser [13] that allmost all graph are of diameter 2. Moreover we have

Lemma 1.2. *if G is a graph of diameter at most 2, then G is regular if and only if G is transmission regular.*

Theorem 1.3. *If G is a k-regular graph of diameter at most 2. Then*

$$\begin{aligned} trace(A^2) &= nk, \\ trace(D^2) &= n(4n - 3k - 4) = 2W(G) + 2n(n - k - 1). \end{aligned}$$

2 Transmission irregular graphs

An automorphism of a graph preserves the distance function. Hence, if u and v are vertices of a graph G such that $\alpha(u) = v$ holds for some $\alpha \in Aut(G)$, then $Tr(u) = Tr(v)$. It follows that a transmission irregular graph is asymmetric and, as it well known, almost all graphs are asymmetric [7]. On the other hand, the fraction of transmission irregular graphs among asymmetric graphs is small as the next result asserts.

Theorem 2.1. *Almost all graphs are not transmission irregular.*

Recently transmission irregular graphs has been interesting and several infinite family of such graphs were constructed. For instance TI starlike trees with three branches characterized in [1,2]. Moreover Dobrynin constructed several infinite family of TI trees of even order [3] 2-connected TI graphs [4,5] and 3-connected TI graphs [6]. The TI star like $T_{n_1 n_2 n_3}$, which $1 \leq n_1 \leq n_2 \leq n_3$ introduced in [1] has vertex set $\{u\} \cup \{x_1, x_2, \cdots x_{n_1}\} \cup \{y_1, y_2, \cdots y_{n_2}\} \cup \{z_1, z_2, \cdots z_{n_3}\}$ and the edge set$\{ux_1, x_1x_2, \cdots x_{n_1-1}x_{n_1}\} \cup \{uy_1, y_1y_2, \cdots y_{n_2-1}y_{n_2}\} \cup \{uz_1, z_1z_2, \cdots z_{n_3-1}z_{n_3}\}$ TI star like

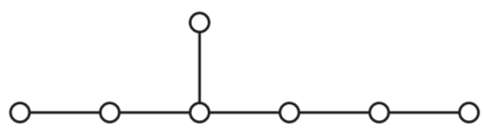

Figure 1: TI tree $T_{1,2,3}$

with three branches were determined in [1] as follows:

Theorem 2.2. *If $1 = n_1 < n_2 < n_3$ then T_{1,n_2,n_3} is transmission regular graph if and only if $n_3 = n_2 + 1$ and $n_2 \notin \{\frac{k^2-1}{2}, \frac{k^2-2}{2}\}$ for some integer $k \geq 3$*

We conclude the section with the following result that in a way support the theorem 2.2.

Theorem 2.3. *If a graph G has three vertices of the same degree, then not both G and $\bar{G}$ are transmission irregular.*

Amog asymmetric trees on same order n, trees $T_{1,2,n-4}$ get the maximum Wiener index.

Theorem 2.4. *Let T be a tree of order n. Then*

$$W(T) \leq \frac{1}{6}(n^3 - 13n + 48)$$

with equality holding if and only if $T = T_{1,2,n-4}$.

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