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A combined scalarizing method in multiobjective problems

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ABSTRACT

In this paper addresses a new scalarization technique for solving multiobjective optimization problems. Theorems are provided on the relation of (weakly, properly) efficient solutions of the multiobjective optimization problem and optimal solutions of the proposed scalarized problems. All the provided results are established with no convexity assumption.

Keywords : Multiobjective optimization, proper efficiency, scalarization.

1. INTRODUCTION

One part of mathematical programming is multiobjective optimization problem when the conflicting objective functions must be minimized or maximized over a feasible set of decisions. Since it is usually not possible to optimize the conflicting functions together, one can only hope to find a trade-off, or compromise, solution. The scalarized problem can then be solved by using standard single-objective optimization techniques. Therefore the scalarized problem can be solved by using standard single objective optimization techniques.

There are many recent publications on applications of MOPs.

Some of the parameter-based scalarization approaches that are widely employed including the weighted-sum method, the ε-constraint method, the normal boundary method and the Pascolett–Serafini method [5, 2-8, 10-12].

Presented an extension of this approach and investigated the relations between approximate optimal solutions of the proposed method and ε-properly efficient solutions. More recently, by including surplus variables in the constraints and penalizing the violations in the objective function of the Pascoletti–Serafini scalarization problem, Akbari et al. [1] presented the flexible Pascoletti–Serafini scalarization method. Moreover, by including slack variables in the constraints of the Pascoletti–Serafini scalarization problem, they obtained necessary and sufficient conditions for proper efficiency.

Gaznavi et al. [9] presented an extension of this approach and investigated the relations between approximate optimal solutions of the proposed method and ε-(properly, weakly) efficient solutions.

The algorithm may solve some redundant PS problems and does not generate a well-spread distribution of non-dominated points in convex MOPs. Burachik et al. [2] proposed the weighted constraint method for solving bi-objective problems that may generate non-Pareto-optimal solutions.

Now, in the present paper, a flexible weighted-constraint scalarization technique, which is applicable for general multiobjective optimization problems is proposed. By this scalarization technique, easy-to-check statements for (weak, proper) efficiency are obtained.

The remainder of this article is organized as follows. in Section 2 some basic definitions and preliminaries are provided. In Section 3, the suggested scalarization approach is described.

In Section 4, relations between optimal solutions of the proposed approach and (weakly, properly) efficient solutions of the related MOP are investigated. The conclusions are derived in Section 7.

2. PRELIMINARIES

Let $X \subseteq R^n$ be a nonempty set and $f: X \to R^p$ with $p \geq 2$ be a vector-valued function. A multiobjective optimization problem may be written as

$$MOP: min f(x) = (f_1(x), \ldots, f_p(x))$$
$$s.t. x \in X.$$

The set of all attainable outcomes, denoted by Y, is defined as the image of X under f. In fact, $Y = f(X) = \{y \in R^m : y = f(x) \text{ for some } x \in X\}$.

The natural ordering cone is defined as follows:

$$R^p_{\geqq} = \{x \in R^p : x_i \geq 0, i = 1, \ldots, p\}.$$

For any $y, \hat{y} \in R^p$:

$y < \hat{y}$ if and only if $y_i < \hat{y}_i \, \forall i = 1, \ldots, p$,
$y \leqq \hat{y}$ if and only if $y_i \leq \hat{y}_i \, \forall i = 1, \ldots, p$,
$y \leq \hat{y}$ if and only if $y_i \leq \hat{y}_i \, \forall i = 1, \ldots, p$ and $y \neq \hat{y}$.

Definition 2.1 A feasible solution $\hat{x} \in X$ is called

(i) an efficient (a Pareto optimal) solution of MOP, if there is no other $x \in X$ such that $f(x) \leq f(\hat{x})$,
(ii) a weakly efficient solution of MOP, if there is no other $x \in X$ such that $f(x) < f(\hat{x})$,
(iii) a strictly efficient solution of MOP, if there is no other $x \in X, x \neq \hat{x}$ such that $f(x) \leqq f(\hat{x})$.

Definition 2.2: A feasible solution $\hat{x} \in X$ is called a properly efficient (a properly Pareto optimal) solution to MOP if it is efficient and there exists a real positive number M such that for each $i \in \{1, \ldots, p\}$ and each $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$, there exists an index $j \in \{1, \ldots, p\}$ with $f_j(\hat{x}) < f_j(x)$ such that

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M.$$

We denote the set of efficient, weakly efficient and properly efficient solutions by X_E, X_{WE} and X_{PE}, respectively. The images of efficient, weakly efficient and properly efficient solutions in the image space R^p are called nondominated, weakly nondominated and properly nondominated solutions and are denoted by Y_N, Y_{WN} and Y_{PN}, respectively.

A single objective optimization problem is demonstrated as follows:

$$SO: min\, g(x)$$
$$s.t. x \in X,$$

where $g: X \to R$.

Definition 2.3 : A feasible solution $\hat{x} \in X$ is said to be

(i) an optimal solution of Problem (SO), if $g(\hat{x}) \leq g(x)$ for all $x \in X$,
(ii) a strictly optimal solution of Problem (SO), if $g(\hat{x}) < g(x)$ for all $x \in X$.

3. THE FLEXIBLE WEIGHTED-CONSTRAINT SCALARIZATION METHOD

Let be given parameters. Burachik et al. [2] proposed the following scalar optimization problem, called the weighted-constraint technique for generate an approximation of the Pareto front in multi-objective problem:

$$P_w^k : \min w_k f_k(x)$$
$$s.t.$$
$$w_i f_i(x) \leq w_k f_k(x), \forall i \neq k$$
$$x \in X.$$

By the scalarized problem P_w^k, sufficient conditions for (properly, weakly) efficient solutions of MOP cannot be obtained. Therefore, an extension of the direction scalarization problem P_w^k is introduced.

Let $w \in R^p_{\geqq}$ and $\sum_{i=1}^{p} w_i = 1$. The flexible weighted-constraint scalarization problem is formulated as follows:

$$FWC : \min w_k f_k(x) + \sum_{\substack{i=1 \\ i \neq k}}^{p} \mu_i s_i$$
$$s.t.$$
$$wf(x) - s - \max_{x \in X} \{w_k f_k(x)\} \in R^p_{\geqq},$$
$$x \in X, s \in R^p_{\geqq}, \mu \in R^p_{\geqq},$$

Where $\mu_i \geq 0$ are nonnegative weights.

4. Characterizing (weakly, properly) efficient solutions

In this section, based on the scalarized problem (FWC), sufficient conditions for obtaining (weakly) efficient solutions and properly efficient solutions of the MOP are provided and necessary and sufficient conditions are obtained for efficient solutions. The following theorem provides a sufficient condition for weak efficiency utilizing the scalarized problem (FWC).

Theorem 4.1 Let $(\hat{x}, \hat{s})$ be an optimal solution of the scalarized problem (FWC). If $\mu \geqq 0$ and $w \geqq 0$ then $\hat{x}$ is a weakly efficient solution of the MOP.

Proof Suppose that $\hat{x}$ is not weakly efficient. Then, there is some $x \in X$ such that $f_i(x) < f_i(\hat{x}), \forall i = 1,2,\ldots, p$. Since $w \geqq 0$, we can write $w_i f_i(x) < w_i f_i(\hat{x}), \forall i \neq k$. Since $(\hat{x}, \hat{s})$ is an optimal solution of (NM), then $w_i f_i(\hat{x}) - \hat{s}_i \leq \max_{x \in X} \{w_k f_k(x)\}, \forall i \neq k$, so $w_i f_i(\hat{x}) - \hat{s}_i \leq \max_{x \in X} \{w_k f_k(x)\}, \forall i \neq k$ then $(x, \hat{s})$ is feasible for (FWC) with an objective value that is smaller than that of $(\hat{x}, \hat{s})$. This contradicts the optimality of $(\hat{x}, \hat{s})$.

Under the uniqueness assumption of the optimal solutions, the following stronger result is obtained for efficiency.

Theorem 4.2 Let $(\hat{x}, \hat{s})$ be an optimal solution of the scalarized problem (FWC) with $\mu \geqq 0$, $w \geqq 0$ and $\hat{x}$ is unique, then $\hat{x}$ is a strictly efficient solution of the MOP.

Proof Assume that x is such that $f(x) \leqq f(\hat{x})$. So, $(x, \hat{s})$ is a feasible solution of (FWC) with $w_k f_k(x) + \sum_{i \neq k} \mu_i \hat{s}_i \leq w_k f_k(\hat{x}) + \sum_{i \neq k} \mu_i \hat{s}_i$, uniqueness of the optimal solution implies that $x = \hat{x}$ .Therefore $\hat{x}$ is a strictly efficient solution of the MOP.

In the following theorem, utilizing the scalarized problem (FWC), a sufficient condition is obtained for efficient solutions of the MOP.

Theorem 43 Let $(\hat{x},\hat{s})$ be an optimal solution of the scalarized problem (FWC). If $\mu>0$, $w>0$ and $\hat{s}>0$, then $\hat{x}$ is an efficient solution of the MOP.

Proof Suppose to the contrary that $\hat{x}$ is not an efficient solution of the MOP. So, there exists a feasible solution $x\in X$ such that

$$f_i(x)\le f_i(\hat{x}),\forall i=1,2,\dots,p,$$

and for some $j\in\{1,\dots,p\}$

$$f_j(x)<f_j(\hat{x}).$$

Therefore, we have

$$w_i f_i(x)-\hat{s}_i\le \max_{x\in X}\{w_k f_k(x)\},\forall i\ne j,k,$$

and

$$w_j f_j(x)-\hat{s}_j<\max_{x\in X}\{w_k f_k(x)\}.$$

Without loss of generality, we can assume $j\ne k$, so we define

$$s_i:=\begin{cases}\hat{s}_i-\alpha & if\ i=j\\ \hat{s}_i & if\ i\ne j\end{cases},\forall i\in\{1,\dots,p\}\setminus\{k\}.$$

Such that $\hat{s}_j-\alpha>0$ and $w_j f_j(x)-\hat{s}_j+\alpha\le \max_{x\in X}\{w_k f_k(x)\}$. Therefore, (x,s) is feasible for (FWC) with $w_k f_k(x)+\sum_{i\ne k}\mu_i s_i\le w_k f_k(\hat{x})+\sum_{i\ne k}\mu_i\hat{s}_i$. This contradicts the optimality of $(\hat{x},\hat{s})$.

In the next theorem, the relationship between optimal solutions of the scalarized problem (FWC) and properly efficient solutions of the MOP is investigated.

Theorem 4.4 Let $\hat{x}$ be an efficient solution of the MOP. Then, there exist $\mu\geqq 0$, $w\geqq 0$ and $\hat{s}\geqq 0$ such that $(\hat{x},\hat{s})$ is an optimal solution (FWC) for all $k\in\{1,2,\dots,p\}$ and $w_i f_i(\hat{x})-\hat{s}_i=\max_{x\in X}\{w_k f_k(x)\},\forall i\ne k$.

Proof Set $\mu_i=\infty,\forall i\ne k$ and $w\geqq 0$. Since $w_j f_j(\hat{x})-\hat{s}_j=\max_{x\in X}\{w_k f_k(x)\},\forall i\ne k$. So $(\hat{x},\hat{s})$ is a feasible of (FWC). We claim that $(\hat{x},\hat{s})$ is also an optimal solution of (FWC). Assume that there is a feasible solution (x,s) for (FWC) with $\mu_i=\infty,\forall i\ne k$ and $w\geqq 0$ such that

$$w_k f_k(\hat{x})+\sum_{i\ne k}\mu_i\hat{s}_i>w_k f_k(x)+\sum_{i\ne k}\mu_i s_i,$$

$$f_k(x)<f_k(\hat{x}). \tag{1}$$

Since (x,s) is a feasible solution of (FWC), so we have

$$w_i f_i(x)-s_i\le \max_{x\in X}\{w_k f_k(x)\}=w_i f_i(\hat{x})-\hat{s}_i,\forall i\ne k$$

$$w_i f_i(x)\le w_i f_i(\hat{x}),\forall i\ne k$$

Since $w>0$, we have

$$f_i(x)\le f_i(\hat{x}),\forall i\ne k \tag{2}$$

According to relations (1) and (2) we have

$$f(x) \le f(\hat{x}),$$

a contraction to $\hat{x}$ being an efficient solution of the MOP.

Next, we state an easy approach to check the sufficient condition for identifying properly efficient solutions among the solutions of (FWC). For the proof we need a technical lemma relating properly efficient solutions of the MOP with the feasible set of (FWC) and the set X, respectively. This lemma is very similar to the idea in Ehrgott and Ruzika (2008).

Lemma 4.1 Let $\hat{x}$ be a properly efficient solution of the MOP with feasible set $X_M = \left\{ x \in X : w_i f_i(\hat{x}) \le \max_{x \in X} \{ w_k f_k(x) \}, \forall i = 1, \dots, p \right\}$. Then $\hat{x}$ is a properly efficient solution of the MOP with feasible set X.

Proof Suppose that $\hat{x}$ is not a properly efficient solution. Then consider a sequence $\{M_\alpha\}$ with $M_\alpha > 0$ and $\lim_{\alpha \to \infty} M_\alpha = \infty$.

For any M_α, there is $x_\alpha \in X$ and an index i with $f_i(x_\alpha) < f_i(\hat{x})$ such that for all $j \ne i$ with $f_j(\hat{x}) < f_j(x_\alpha)$, we have

$$\frac{f_i(\hat{x}) - f_i(x_\alpha)}{f_j(x_\alpha) - f_j(\hat{x})} > M_\alpha. \qquad (1)$$

We choose a subsequence of $\{M_\alpha\}$ such that index i is fixed for each α. We assume that for each α $J = \{ j \in \{1, \dots, p\} : f_j(\hat{x}) < f_j(x_\alpha) \}$ is constant. According to relation (1) and since $f(X)$ is bounded, we have

$$\lim_{\alpha \to \infty} f_j(x_\alpha) = f_j(\hat{x}).$$

So, there is $\alpha_0 > 0$ such that

$$w_j f_j(x_\alpha) < \max_{x \in X} \{ w_k f_k(x) \} + \hat{s}_j, \forall \alpha > \alpha_0, \forall j \in J.$$

Thus, $\{x_\alpha\}_{\alpha > \alpha_0} \subseteq X_M$. This contradicts the proper efficiency of $\hat{x}$ for the MOP with feasible set X_M.

Theorem 4.5 Let $(\hat{x}, \hat{s})$ be an optimal solution of the scalarized problem (FWC). If $\mu > 0$, $w > 0$ and $\hat{s} > 0$, then $\hat{x}$ is an properly efficient solution of the MOP.

Proof According to Theorem 2.3 $\hat{x}$ is an efficient solution of the MOP for any $\mu > 0$ and $w > 0$. Since $(\hat{x}, \hat{s})$ is an optimal solution of the scalarized problem (FWC), we have

$$w_k f_k(\hat{x}) + \sum_{i \ne k} \mu_i \hat{s}_i = w_k f_k(\hat{x}) + \sum_{i \ne k} \mu_i \left(w_i f_i(\hat{x}) - \max_{x \in X} \{ w_k f_k(x) \} \right).$$

So, is an optimal solution of the weighted sum problem

$$\min \left\{ \sum_{i=1}^{p} \mu_i w_i f_i(\hat{x}) : \mu_i w_i = 1, w_i f_i(\hat{x}) \le \max_{x \in X} \{ w_k f_k(x) \} \right\}.$$

According to Geoffrion's theorem, $\hat{x}$ is a properly efficient solution of the MOP with additional constraints. By lemma 2.1 $\hat{x}$ is a properly efficient solution of the MOP.

5. CONCLUSIONS

We have proposed a new scalarization technique for solving multiobjective optimization has been proposed. Using the proposed approach, necessary and sufficient conditions for (weakly, properly) efficient solutions were established.

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