DIMENSION OF $AB5^*$ MODULES

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ABSTRACT. In this article, the α-shortness of R-modules in the category of $AB5^*$ were studied. We proved that if M is an R-module in $AB5^*$ which has Noetherian dimension and $\{N_i\}$ is a family of submodules of M such that $n\text{-dim}\,\frac{M}{N_i} \leq \alpha$, for each i, then $n\text{-dim}\,\frac{M}{\cap N_i} \leq \alpha$ (this is the duall of a famous result due to Lenegan). Using this, we give an structure theorem for α-short modules in $AB5^*$ and finally, we classify α-short modules in this category.

1. Introduction

The concepts of α-short modules and α-Krull modules have studied and investigated respectively in [1, 2]. It is proved that if M is an α-short module, then either $n\text{-dim}\,M = \alpha$ or $n\text{-dim}\,M = \alpha + 1$. Similarely, if M is an α-Krull module, then either $k\text{-dim}\,M = \alpha$ or $k\text{-dim}\,M = \alpha + 1$. In particular, it is proved that a semiprime non-division-ring R is α-short if and only if $n\text{-dim}\,R = \alpha$, where $\alpha \geq 0$, see [1, Proposition 2.18]. From this, it naturally raised the question of characterization the R-modules M that are α-short if and only if $n\text{-dim}\,M = \alpha$. We note that in [8, Theoerem 5.2] we partially answered to this question. We also classified α-Krull modules in [8]. Motivated by this, in this paper, we classify the α-short modules in the category of $AB5^*$. For this purpose, we proved the dual of [9, 6.2.17] (that is known as Lenegan's Lemma) for Noetherian dimension of $AB5^*$ modules. Then we give an structure theorem for α-short modules in this category and using that, we classify the α-short $AB5^*$ modules.

2. Noetherian Dimension of $AB5^*$ modules

We begin with well-known definition of $AB5^*$ modules.

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Definition 2.1. An R-module M is said to satisfies the property $AB5^*$, if $B+\bigcap_{i\in I} A_i = \bigcap_{i\in I}(B+A_i)$, for every submodule B and chain of submodules $\{A_i\}_{i\in I}$.

Note that most modules M dont have this property. Fortunately, Artinian modules and linearly compact modules satisfy $AB5^*$ property, see [12, 29.8].

Lemma 2.2. *let M be an R-module with Krull dimension and $\{N_i\}_{i\in\Delta}$ be a family of the submodules of M which k-dim $N_i \leq \alpha$, for each $i \in \Delta$. If $M = \Sigma_{i\in\Delta} N_i$, then k-dim $M \leq \alpha$.*

The proof of the above lemma is due to T. H. Lenegan. We are going to prove the duall of this result for the Noetherian dimension of modules. The following two results are well-known.

Theorem 2.3. *An R-module M has Krull dimension if and only if it has Noetherian dimension.*

Lemma 2.4. *Suppose that a module M has infinite chains*

$$M = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$$

and

$$0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$$

such that, for each $n \geq 0$, $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$. Then, M does not have a Krull dimension.

The following is the dual of Lenegan Lemma for Noetherian dimension of modules in the category of $AB5^*$ modules.

Theorem 2.5. *Let M be an R-module which satisfies in the property of $AB5^*$ and has Noetherian dimension and let $\{N_i\}$ a family of submodules of M such that n-dim $\frac{M}{N_i} \leq \alpha$, for each i. Then n-dim $\frac{M}{\cap N_i} \leq \alpha$.*

Proof. Set $N = \cap N_i$ and let $0 \neq \frac{A}{N}$ be a non-zero submodule of $\frac{M}{N}$, then $N \subsetneq A$, so there exists an indexed submodule, N_1 say, such that $A \nsubseteq N_1$. Thus $N \subseteq A \cap N_1 \subsetneq A$ and hence

$$0 \neq \frac{A}{A \cap N_1} \cong \frac{A+N_1}{N_1}$$

and n-dim $\frac{A}{A\cap N_1} \leq \alpha$. This shows that each submodule A of M, properly containing N, contains a proper submodule A' containing N and n-dim $\frac{A}{A'} \leq \alpha$. Thus, there exists a descending chain

$$M = A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \dots N$$

with ordinal indexes such that for each ordinal λ, n-dim $\frac{A_\lambda}{A_{\lambda+1}} \leq \alpha$. For limit ordinal λ, define $A_\lambda = \cap_{\delta<\lambda} A_\delta$. Since M is a set, there exists an ordinal number λ such that $A_\lambda = N$. Now we proceed by induction on λ to show that

$$P(\lambda) : n\text{-dim}\, \frac{M}{A_\lambda} \leq \alpha.$$

Clearly $P(0)$ and also $P(\lambda)$, for every non-limit ordinal number λ, since n-dim $\frac{M}{A_\lambda} = sup\{n\text{-dim}\, \frac{M}{A_{\lambda-1}}, n\text{-dim}\, \frac{A_{\lambda-1}}{A_\lambda}\}$. Now let λ be a limit ordinal number. If n-dim $\frac{M}{A_\lambda} \nleq \alpha$, then there exists an ascending chain

$$A_\lambda = N = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \dots$$

such that n-dim $\frac{B_{i+1}}{B_i} \geq \alpha$. Note that $B_1 \nsubseteq N$, so there exists $\delta < \lambda$ such that $B_1 \nsubseteq A_\delta$. Set $A_\delta = A_1$, then $B_1 \nsubseteq A_1$ and so $A_0 \cap B_1 \nsubseteq A_1 + B_0$. Suppose that $A_i \cap B_{i+1} \nsubseteq A_{i+1} + B_i$, for each $0 \leq i \leq n-1 < \lambda$. In particular, $A_{n-1} \cap B_n \nsubseteq A_n + B_{n-1}$. We seek A_{n+1} and B_{n+1} such that $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$. If for each $k > 0$, $A_n \cap B_{n+k} \subseteq B_n$, then $A_n \cap B_{n+k} \subseteq B_{n+k-1}$. Then by modular laws, it follows that that $B_{n+k-1} = (A_n \cap B_{n+k}) + B_{n+k-1} = (A_n + B_{n+k-1}) \cap B_{n+k}$. Therefore

$$\frac{B_{n+k}}{B_{n+k-1}} \cong \frac{B_{n+k}}{B_{n+k} \cap (A_n + B_{n+k-1})} \cong \frac{A_n + B_{n+k}}{A_n + B_{n+k-1}} \cong \frac{\frac{A_n + B_{n+k}}{A_n}}{\frac{A_n + B_{n+k-1}}{A_n}}$$

This shows that n-dim $\frac{\frac{A_n + B_{n+k}}{A_n}}{\frac{A_n + B_{n+k-1}}{A_n}} \geq \alpha$. Hence

$$\frac{A_n + B_n}{A_n} \subseteq \frac{A_n + B_{n+1}}{A_n} \subseteq \frac{A_n + B_{n+2}}{A_n} \subseteq \dots$$

is an ascending chain in $\frac{M}{A_n}$ such that every quotient has Noetherian dimension at least α, contradiction to n-dim $\frac{M}{A_n} \leq \alpha$. Consequently, there exists $k > 0$ such that $A_n \cap B_{n+k} \nsubseteq B_n$. Set $B_{n+k} = B_{n+1}$ and conclude $A_n \cap B_{n+1} \nsubseteq B_n$. But $B_n = B_n + N = B_n + A_\lambda$, so $A_n \cap B_{n+1} \nsubseteq B_n + \cap_{\delta<\lambda} A_\delta$. Since M has the property of $AB5^*$, we have $B_n + \cap_{\delta<\lambda} A_\delta = \cap_{\delta<\lambda}(B_n + A_\delta)$, consequently there exists $\delta > n$ that $A_n \cap B_{n+1} \nsubseteq A_\delta + B_n$. Set $A_\delta = A_{n+1}$ and conclude that $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$. Therefore there exist the cahins

$$N = B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 \subseteq ...$$

and

$$M = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$$

such that $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$. The Lemma 2.4 shows that M has no Krull dimension and so by Theorem 2.3 it has no Noetherian dimension too, a contradiction. Therefore $n\text{-dim}\, \frac{M}{N} \leq \alpha$ and this complete the proof. □

3. α-short $AB5^*$ modules

Definition 3.1. Let $\alpha \geq -1$ be an ordinal number. An R-module M is called to be α-short if for every submodule N of M either $n\text{-dim}\, N \leq \alpha$ or $n\text{-dim}\, \frac{M}{N} \leq \alpha$ and α is the least ordinal with this property.

It is easy to see that -1-short modules are just simple modules and 0-short modules are just short modules.

Lemma 3.2. *If M is an α-short module, then any submodule and any factor module of M is β-short for some ordinal $\beta \leq \alpha$.*

Theorem 3.3. *If M is an α-short R-module, then either $n\text{-dim}\, M = \alpha$ or $n\text{-dim}\, M = \alpha + 1$.*

Theorem 3.4. *Let N be a submodule of an R-module M:*

(1) *If N is α-short and $n\text{-dim}\, \frac{M}{N} \leq \alpha$, then M is α-short.*
(2) *If $\frac{M}{N}$ is α-short and $n\text{-dim}\, N \leq \alpha$, then M is α-short.*

Proposition 3.5. *Let R be a semiprime ring and $\alpha \geq 0$ be an ordinal number. Then R is α-short if and only if $n\text{-dim}\, R = \alpha$.*

We use the following notations throughout the paper.

(1) $\mathcal{M}$ denotes the set of all modules with Krull and Noetherian dimension.
(2) $\mathcal{M}_\alpha$ donetes the set of all α-short modules in $\mathcal{M}$.
(3) $\mathcal{A}_\alpha =$ The set of all α-short modules $M \in \mathcal{M}_\alpha$ with $n\text{-dim}\, M = \alpha$ and $\mathcal{A} = \cup_\alpha \mathcal{A}_\alpha$.
(4) $\mathcal{B}_\alpha =$ The set of all α-short modules $M \in \mathcal{M}_\alpha$ with $n\text{-dim}\, M = \alpha + 1$ and $\mathcal{B} = \cup_\alpha \mathcal{B}_\alpha$.

Remark 3.6. It is easy to see that $\mathcal{A}_\alpha \cap \mathcal{B}_\alpha = \emptyset$ and $\mathcal{A}_\alpha \cup \mathcal{B}_\alpha = \mathcal{M}_\alpha$ and so $\{\mathcal{A}_\alpha, \mathcal{B}_\alpha\}$ is a partition of $\mathcal{M}_\alpha$. Similarly, $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $\mathcal{M} = \mathcal{A} \cup \mathcal{B}$ so $\{\mathcal{A}, \mathcal{B}\}$ is a partition of $\mathcal{M}$.

The following result is our structure theorem for α-short modules in the catagory of $AB5^*$.

Theorem 3.7. *Let M be an R-module in the category of $AB5^*$ and α be an ordinal number. The following are equivalent.*

1. *M is an α-short module.*
2. *There exists a submodule $A(\alpha)$ of M such that n-$\dim \frac{M}{A(\alpha)} \leq \alpha$ and n-$\dim B \leq \alpha$ for any submodule $B \nsupseteq A(\alpha)$ and α is the least ordinal with this property.*

Proof. Let M be α-short and $\Sigma = \{X \leq M : n\text{-}\dim \frac{M}{X} \leq \alpha\}$. Clearly $M \in \Sigma$ and so $\Sigma \neq \emptyset$. Set $A(\alpha) = \cap_{X \in \Sigma} X$, then n-$\dim \frac{M}{A(\alpha)} \leq \alpha$ by Theorem 2.5. Now let B be a submodule of M such that $A(\alpha) \nsubseteq B$, then $B \notin \Sigma$ and so n-$\dim \frac{M}{B} \nleq \alpha$. Hence n-$\dim B \leq \alpha$, since M is α-short. To show that α is the least ordinal with this property, it suficies to prove the converse. Conversly, let M has a submodule $A(\alpha)$ with mentioned conditions and let $B \leq M$. If $A(\alpha) \subseteq B$, then n-$\dim \frac{M}{B} \leq n$-$\dim \frac{M}{A(\alpha)} \leq \alpha$. If $A(\alpha) \nsubseteq B$, then n-$\dim B \leq \alpha$, but α is the least ordinal with this property and so M is α-short. □

Theorem 3.8. *Let M be an R-module. If $M \in \mathcal{B}_\alpha$ and $N \leq M$, then either $N \in \mathcal{B}_\alpha$ or $\frac{M}{N} \in \mathcal{B}_\alpha$.*

Proof. By the above notations, M is α-short and n-$\dim M = \alpha + 1$. We have the following cases.

1. If n-$\dim N > n$-$\dim \frac{M}{N}$, then n-$\dim N = n$-$\dim M = \alpha + 1$. Also N is β-Krull for some $\beta \leq \alpha$. If $\beta < \alpha$, by Theorem 3.3, we have n-$\dim N \leq \beta + 1 \leq \alpha$, a contradiction. Consequently, N is α-Krull with n-$\dim N = \alpha + 1$. Therefore $N \in \mathcal{B}_\alpha$.
2. If n-$\dim \frac{M}{N} > n$-$\dim N$, then n-$\dim \frac{M}{N} = n$-$\dim M = \alpha + 1$. Also $\frac{M}{N}$ is β-Krull for some $\beta \leq \alpha$. If $\beta < \alpha$, by Theorem 3.3, we have k-$\dim \frac{M}{N} \leq \beta + 1 \leq \alpha$, a contradiction. Consequently, $\frac{M}{N}$ is α-Krull with n-$\dim \frac{M}{N} = \alpha + 1$. Therefore $\frac{M}{N} \in \mathcal{B}_\alpha$.
   At last,
3. If n-$\dim N = n$-$\dim \frac{M}{N}$, then the same argument shows that both N and $\frac{M}{N}$ belong to $\mathcal{B}_\alpha$.

□

In view of the proof of Theorem 3.8 the next corollary is now immediate.

Corollary 3.9. *Let $M \in \mathcal{B}_\alpha$ and $N \leq M$, then :*

1. *If n-$\dim N > n$-$\dim \frac{M}{N}$, then $N \in \mathcal{B}_\alpha$.*
2. *If n-$\dim N < n$-$\dim \frac{M}{N}$, then $\frac{M}{N} \in \mathcal{B}_\alpha$.*
3. *If n-$\dim N = n$-$\dim \frac{M}{N}$, then $N, \frac{M}{N} \in \mathcal{B}_\alpha$.*

Corollary 3.10. *If N is a submodule of M such that $N, \frac{M}{N} \in \mathcal{A}$, then $M \in \mathcal{A}$.*

Corollary 3.11. *If $M_1, M_2, ..., M_n \in \mathcal{A}$, then $M_1 \oplus M_2 \oplus ... \oplus M_n \in \mathcal{A}$.*

Remark 3.12. The converse of Corolary 3.11 is not true in general . For example if $M = M_1 \oplus M_2$ where M_1 and M_2 are simple modules, then $M_1, M_2 \in \mathcal{B}_{-1}$ but $M \in \mathcal{A}$ becausee of semisimplisity. Note that any semisimple module $M \in \mathcal{M}$ is short and Noetherian.

Corollary 3.13. *If $M \in \mathcal{B}_\alpha$ for some ordinal number $\alpha \geq 0$, then M has an essential submodule $E \leq_e M$ such that $\frac{M}{E} \in \mathcal{B}_\alpha$*

Proof. Since $M \in \mathcal{B}_\alpha$ so $n\text{-dim}\, M = \alpha + 1 > 0$, thus M is not semisimple (note that semisimple modules with krull dimension are both Artinian and Noetherian). This implies that M has proper essential submodule. By Lemma 2.4, $n\text{-dim}\, M = sup\{n\text{-dim}\, \frac{M}{E} : E \leq_e M\}$, so $n\text{-dim}\, \frac{M}{E} = \alpha + 1$ for some essential submodule E of M. This implies that $\frac{M}{E} \in \mathcal{B}_\alpha$ by theorem 3.4. □

Recall that for any ordinal number α, by an α-atomic modue we means an R-module M such that $n\text{-dim}\, M = \alpha$ and $ndim\, N < \alpha$ for every proper submodule N of M.

Theorem 3.14. *Let M be an α-short R-module in the category $AB5^*$. The following statements are equivalent.*

1. *$M \in \mathcal{B}_\alpha$.*
2. *The submodule $A(\alpha)$ of M is $(\alpha + 1)$-atomic.*

Proof. If $M \in \mathcal{B}_\alpha$, then $n\text{-dim}\, M = \alpha + 1$ but $n\text{-dim}\, \frac{M}{A(\alpha)} \leq \alpha$ this implies that $A(\alpha) \neq 0$ and $n\text{-dim}\, A(\alpha) = \alpha + 1$, now if $B \subsetneq A(\alpha)$, then $n\text{-dim}\, B \leq \alpha$ by Theorem 3.7 and so $A(\alpha)$ is $(\alpha + 1)$-atomic. Conversely if $A(\alpha)$ is $(\alpha + 1)$-atomic, then $A(\alpha)$ is α-short and by Theorem 3.7, $n\text{-dim}\, \frac{M}{A(\alpha)} \leq \alpha$, so M is α-short, by Theorem 3.4. □

The next theorem determines all modules in the category of $AB5^*$, belong to $\mathcal{B}$.

Theorem 3.15. *The following statements are equivalent for any module M in the category of $AB5^*$:*

1. *$M \in \mathcal{B}$*
2. *$n\text{-dim}\, M$ is not a limit ordinal and M has an atomic submodule A with $n\text{-dim}\, A > n\text{-dim}\, \frac{M}{A}$.*

Proof. If $M \in \mathcal{B}$, then there exists ordinal number α such that $M \in \mathcal{B}_\alpha$ so $n\text{-}\dim M = \alpha+1$ is not a limit ordinal and $A(\alpha)$ is an $(\alpha+1)$-atomic submodule of M by Theorem 3.14. Moreover, $n\text{-}\dim \frac{M}{A(\alpha)} \leq \alpha < \alpha + 1 = n\text{-}\dim A(\alpha)$ by theorem 3.7. Conversely let $n\text{-}\dim M = \alpha + 1$ and A be an atomic submodule of M such that $n\text{-}\dim A > n\text{-}\dim \frac{M}{A}$, then $n\text{-}\dim A = \alpha + 1$, but A is α-short and so by theorem 3.4, M is α-short too, therfore $M \in \mathcal{B}$.

□

The next important result is now immediate.

Theorem 3.16. *The following stetements are equivalent for any module M in the category of $AB5^*$:*

1. $M \in \mathcal{A}$.
2. $n\text{-}\dim M$ *is a limit ordinal or* $n\text{-}\dim A \leq n\text{-}\dim \frac{M}{A}$, *for any atomic submodule A of M.*

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