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Inclusion and neighborhood of multivalent q function with negative coefficients

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Abstract. This article presents certain families of analytic and multivalent functions regarding q starlikeness and q convexity of order α. By using concept of neighborhoods of p valent function, we prove the coefficient bound and associated inclusion relation for the (δ, q)-neighborhood of multivalent function with negative coefficient defined by q-derivative.
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1. Introduction

Let $\mathcal{U}_p$ $(p \in \mathbb{N} = \{1, 2, ...\})$ contain all multivalent functions say f that are holomorphic or analytic in a subset $D = \{z; |z| < 1\}$ of a complex plan $\mathbb{C}$ and having the series form:

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \ (a_{k+p} \geq 0, z \in D). \tag{1}$$

In particular, if $p = 1$ the class $\mathcal{U}$ of univalent function was investigated by Graham and Kohr [3]. Some important elementary concept details and definitions of the q calculus which play vital role in our presentation will be recalled in [1].

For given $q \in (0, 1)$, the derivative in q-analogue of f is given by

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \ (z \neq 0, q \neq 1). \tag{2}$$

We note that the q-derivative operator D_q converges to the ordinary derivative operator as follows:

$$\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z).$$

Making (1) and (2), we easily get that for $n \in \mathbb{N}$ and $z \in D$:

$$D_q f(z) = [p]_q z^{p-1} + \sum_{k=1}^{\infty} [k+p]_q a_{k+p} z^{k+p-1},$$

where the q derivative operator $D_q f(z)$ of function f is defined by

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & (z \neq 0; 0 < q < 1), \\ f'(0), & z = 0. \end{cases}$$

provided that $f'(0)$ exists, and the number $[n]_q$ is

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{k=1}^{n-1} q^k, \ [0]_q = 0.$$

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The class denoted by $S_q^*(p,\alpha)$ of p-valent q-starlike function that of order α defined as

$$S_q^*(p,\alpha)=\left\{f\in\mathcal{U}_p : Re\left(\frac{zD_qf(z)}{f(z)}\right)>\alpha,(z\in D,0\leq\alpha<1)\right\}.$$

Also, the class $C_q(p,\alpha)$ of p-valent q-convex functions of order α is defined as:

$$C_q(p,\alpha)=\left\{f\in\mathcal{U}_p : Re\left(1+\frac{D_q(zD_qf(z))}{D_qf(z)}\right)>\alpha,(z\in D,0\leq\alpha<1)\right\}.$$

In its special case when $p=1$, the classes $S_q^*(\alpha)$ and $C_q(\alpha)$ of q-starlike functions of order α and q-convex functions of order α were studied in earlier work (see [5]).

In [2, 4] defined the (n,δ)-neighborhood of a function $f(z)\in\mathcal{U}_p$. Now, we define (δ,q)-neighborhood of a function $f(z)\in\mathcal{U}_p$ as

$$N_{\delta,q}(f)=\left\{g\in\mathcal{U}_p : g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \text{ and } \sum_{k=n+p}^{\infty}[k]_q|a_k-b_k|\leq\delta\right\},\quad(\delta>0). \tag{3}$$

It follow from (3) that, if $h(z)=z^p$, $(p\in\mathbb{N}$ then

$$N_{\delta,q}(h)=\left\{g\in\mathcal{U}_p : g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \text{ and } \sum_{k=n+p}^{\infty}[k]_q|b_k|\leq\delta\right\},\quad(\delta>0). \tag{4}$$

The object of the present paper is to investigate the various properties and characteristics of analytic p-valent functions belonging to the class $T_q(p,\lambda,\alpha)$, which includes functions $f(z)$ satisfying the following inequality

$$Re\left\{\frac{zD_qf(z)+\lambda zD_q(zD_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}\right\}>\alpha,(z\in D,0\leq\alpha<1),$$

which have introduced here. Clearly, we have

$$T_q(p,0,\alpha)=S_q^*(p,\alpha)\qquad \text{and} \qquad T_q(p,1,\alpha)=C_q(p,\alpha),$$

in terms of the simpler classes $S_q^*(p,\alpha)$ and $C_q(p,\alpha)$, respectively. Also, let $R_q(p,\lambda,\alpha)$ denote the subclass of $\mathcal{U}_p$ consisting of function $f(z)$ which satisfy the inequality

$$Re\left\{D_qf(z)+\lambda D_q(zD_qf(z))\right\}>0.$$

Apart from deriving a set coefficient bounds for each of these function classes, we establish several inclusion relationships involving the (q,δ) neighborhoods of analytic p-valent functions (with negative and missing coefficient) belonging to this subclasses.

2. A set of coefficient inequality

In this section, we prove the following results which yield the coefficient inequality for functions in the subclasses $T_q(p,\lambda,\alpha)$.

2.1. THEOREM. *A function $f(z)\in\mathcal{U}_p$ is in the class $T_p(p,\lambda,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left(-\alpha(1-\lambda)+(1-\lambda\alpha)[k+p]_q+\lambda[k+p]_q^2\right)a_{k+p}\leq\phi_p, \tag{5}$$

$$(0\leq\alpha<1,0\leq\lambda\leq1,(1-\lambda\phi_p)[p]_q+\lambda[p]_q^2>\phi_p(1-\lambda),z\in D,p\in\mathbb{N}).$$

where $\phi_p=-\alpha(1-\lambda)+(1-\alpha\lambda)[p]_q+\lambda[p]_q^2$. The result is sharp.

PROOF. Suppose that $f(z)\in T_q(p,\lambda,\alpha)$. Then, we have

$$Re\left\{\frac{([p]_q+\lambda[p]_q^2)z^p-\sum_{k=1}^{\infty}[k+p]_q+\lambda[k+p]_q^2)a_{k+p}z^{k+p}}{(1-\lambda+\lambda[p]_q)z^p-\sum_{k=1}^{\infty}(1-\lambda+\lambda[k+p]_q)a_{k+p}z^{k+p}}\right\}>\alpha.$$

$$(0\leq\alpha<1,0\leq\lambda\leq1,(1-\lambda\phi_p)[p]_q+\lambda[p]_q^2>\phi_p(1-\lambda),z\in D,p\in\mathbb{N}).$$

If we chose z to be real and let $z \to 1^-$, we obtain

$$\frac{([p]_q + \lambda[p]_q^2) - \sum_{k=1}^{\infty}[k+p]_q + \lambda[k+p]_q^2)a_{k+p}}{(1-\lambda+\lambda[p]_q) - \sum_{k=1}^{\infty}(1-\lambda+\lambda[k+p]_q)a_{k+p}} > \alpha.$$

$$(0 \le \alpha < 1, 0 \le \lambda \le 1, (1-\lambda\phi_p)[p]_q + \lambda[p]_q^2 > \phi_p(1-\lambda), z \in D, p \in \mathbb{N}).$$

or equivalently

$$-\sum_{k=1}^{\infty}\left(-\alpha(1-\lambda) + (\lambda\alpha - 1)[k+p]_q + \lambda[k+p]_q^2\right)a_{k+p} > \alpha(1-\lambda) - (1-\lambda\alpha)[p]_q - \lambda[p]_q^2.$$

$$(0 \le \alpha < 1, 0 \le \lambda \le 1, (1-\lambda\phi_p)[p]_q + \lambda[p]_q^2 > \phi_p(1-\lambda), z \in D, p \in \mathbb{N}).$$

which is precisely assertion (5) of theorem 2.1.

Conversely, suppose that the inequality (5) holds true and let

$$z \in \partial D = \{z \in \mathbb{C}, |z| = 1\}.$$

Then, we find from the definition (1) that

$$\begin{aligned}
&\left|\frac{zD_qf(z) + \lambda zD_q(zD_qf(z))}{(1-\lambda)f(z) + \lambda zD_qf(z)} - \phi_p\right| \\
&= \frac{\left|(-\phi_p(1-\lambda) + (1-\lambda\phi_p)[p]_q + \lambda[p]_q^2)z^p - \sum_{k=1}^{\infty}\left(-\phi_p(1-\lambda) + (1-\lambda\phi_p)\,[p+k]_q + \lambda[p+k]_q^2\right)a_{k+p}z^{k+p}\right|}{|(1-\lambda+\lambda[p]_q)z^p - \sum_{k=1}^{\infty}(1-\lambda+\lambda[k+p]_q)a_{k+p}z^{k+p}|} \\
&\le \frac{(-\phi_p(1-\lambda) + (1-\lambda\phi_p)[p]_q + \lambda[p]_q^2)|z^p| + \sum_{k=1}^{\infty}\left(-\phi_p(1-\lambda) + (1-\lambda\phi_p)\,[p+k]_q + \lambda[p+k]_q^2\right)a_{k+p}|z^{k+p}|}{(1-\lambda+\lambda[p]_q)|z^p| - \sum_{k=1}^{\infty}(1-\lambda+\lambda[k+p]_q)a_{k+p}|z^{k+p}|} \\
&\le \phi_p - \alpha.
\end{aligned}$$

$$(0 \le \alpha < 1, 0 \le \lambda \le 1, (1-\lambda\phi_p)[p]_q + \lambda[p]_q^2 > \phi_p(1-\lambda), z \in \partial D, p \in \mathbb{N}).$$

provided that the inequality (5) is satisfied. Hence by the maximum modulus theorem, we have $f(z) \in T_q(p,\alpha,\lambda)$.

Finally, we note that the assertion (5) of theorem 2.1 is sharp, the extremal function being

$$f(z) = z^p - \frac{-\alpha(1-\lambda) + (1-\alpha\lambda)[p]_q + \lambda[k+p]_q^2}{-\alpha(1-\lambda) + (1-\alpha\lambda)[p+k]_q + \lambda[k+p]_q^2}z^{k+p}, \qquad (z \in D, 0 \le \alpha < 1).$$

□

2.2. Theorem. *Let $f(z) \in \mathcal{U}_p$ be given by (1). Then $f(z) \in R_q(p,\lambda,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left([k+p]_q + \lambda[k+p]_q^2\right)a_{k+p} \le [p]_q + \lambda[p]_q^2 - \alpha$$

2.3. Theorem. *Let $f(z) \in \mathcal{U}_p$ be in the class $T_q(p,\alpha,\lambda)$. Then*

$$(6) \qquad \sum_{k=1}^{\infty}a_{k+p} \le \frac{\phi_p}{\phi_{k+p}} \quad \text{and} \quad \sum_{k=1}^{\infty}[k+p]_q a_{k+p} \le \frac{\phi_p\phi_{k+p} - \alpha(1-\lambda)}{\phi_{k+p}(\phi_{k+p} + \alpha(1-\lambda))},$$

where $\phi_p = -\alpha(1-\lambda) + (1-\alpha\lambda)[p]_q + \lambda[p]_q^2$.

Proof. By using Theorem 2.1, we find from (5) that

$$\begin{aligned}
-\alpha(1-\lambda) + (1-\alpha\lambda)[k+p]_q + \lambda[k+p]_q^2 \sum_{k=1}^{\infty}a_{k+p} \\
\le \sum_{k=1}^{\infty}(-\alpha(1-\lambda) + (1-\alpha\lambda)[k+p]_q + \lambda[k+p]_q^2)a_{k+p} \\
\le -\alpha(1-\lambda) + (1-\alpha\lambda)[p]_q + \lambda[p]_q^2,
\end{aligned}$$

which immediately yields the first assertion (6) of Theorem 2.3.

By appealing (5), we also have

$$-\alpha(1-\lambda)\sum_{k=1}^{\infty}a_{k+p}+((1-\alpha\lambda)+\lambda[k+p]_q)\sum_{k=1}^{\infty}[k+p]_q a_{k+p}\leq\sum_{k=1}^{\infty}\phi_{k+p}a_{k+p}\leq\phi_p.$$

so that,

$$\begin{aligned}((1-\alpha\lambda)+\lambda[k+p]_q)\sum_{k=1}^{\infty}[k+p]_q a_{k+p} &\leq \phi_p+\alpha(1-\lambda)\sum_{k=1}^{\infty}a_{k+p}\\ &= \phi_p+\alpha(1-\lambda)\frac{\phi_p}{\phi_{k+p}}.\end{aligned}$$

This yields

$$(\phi_{k+p}+\alpha(1-\lambda))\sum_{k=1}^{\infty}[k+p]a_{k+p}\leq\frac{\phi_p\phi_{k+p}-\alpha(1-\lambda)}{\phi_{k+p}},$$

which readily implies the coefficient inequality

$$\sum_{k=1}^{\infty}[k+p]_q a_{k+p}\leq\frac{\phi_p\phi_{k+p}-\alpha(1-\lambda)}{\phi_{k+p}\left(\phi_{k+p}+\alpha(1-\lambda)\right)}.$$

□

2.4. Theorem. *Let $f(z)\in\mathcal{U}_p$ be in the class $R_q(p,\lambda,\alpha)$. Then*

$$\sum_{k=1}^{\infty}a_{k+p}\leq\frac{[p]_q+\lambda[p]_q^2-\alpha}{[k+p]_q+\lambda[k+p]_q^2},\qquad and\qquad \sum_{k=1}^{\infty}[k+p]_q a_{k+p}\leq\frac{[p]_q+\lambda[p]_q^2-\alpha}{1+\lambda[k+p]_q}.$$

3. Inclusion relations Involving the (n,δ)-nighborhoods

In this section, we determine the neighborhood properties for each of the following function classes

$$T_q(p,\alpha,\lambda),\qquad R_q(p,\alpha,\lambda),\qquad S_q^*(p,\alpha),\qquad C_q(p,\alpha).$$

3.1. Theorem. *If $f\in\mathcal{U}_p$ is in the class $T_q(p,\alpha,\lambda)$, then*

$$T_q(p,\alpha,\lambda)\subset N_{\delta,q}(h),\tag{7}$$

where $h(z)=z^p, z\in D$ and

$$\delta:=\frac{\phi_p\phi_{k+p}-\alpha(1-\lambda)}{\phi_{k+p}(\phi_{k+p}+\alpha(1-\lambda))}.\tag{8}$$

Proof. Assertion (13) would follow easily from the definition of $N_{\delta,q}(h,f)$, which is given by (4) with $g(z)$ replaced by $f(z)$, and the second assertion (6) of Theorem 2.3. □

3.2. Theorem. *If $f\in\mathcal{U}_p$ is in the class $R_q(p,\alpha,\lambda)$, then*

$$R_q(p,\alpha,\lambda)\subset N_{\delta,q}(h),\tag{9}$$

where $h(z)=z^p, z\in D$ and

$$\delta:=\frac{[p]_q+\lambda[p]_q^2-\alpha}{1+\lambda[k+p]_q}.\tag{10}$$

Putting $\lambda=0$ in Theorem 3.1, we obtain the following corollary:

3.3. Corollary. *If $f \in \mathcal{U}_p$ is in the class $S_q^*(p,\alpha)$, then*

$$S_q^*(p,\alpha) \subset N_{\delta,q}(h), \tag{11}$$

where $h(z) = z^p, z \in D$ and

$$\delta := \frac{(-\alpha + [p]_q)(-\alpha + [p+k]_q)}{[p+k]_q(-\alpha + [p+k]_q)}. \tag{12}$$

Putting $\lambda = 1$ in Theorem 3.1, we obtain the following corollary:

3.4. Corollary. *If $f \in \mathcal{U}_p$ is in the class $C_q(p,\alpha)$, then*

$$C_q(p,\alpha) \subset N_{\delta,q}(h), \tag{13}$$

where $h(z) = z^p, z \in D$ and

$$\delta := \frac{(1-\alpha)[p]_q + [p]_q^2}{(1-\alpha)[k+p]_q + [k+p]_q^2}. \tag{14}$$

3.5. Definition. A function $f \in \mathcal{U}_p$ is said to be in the class $S_q(p,\alpha,\lambda)$ if there exist $g \in R_q(p,\alpha,\lambda)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \alpha.$$

3.6. Theorem. *If $g \in R_q(p,\alpha,\lambda)$ and*

$$\alpha = 1 - \delta \frac{1 + \lambda[k+p]_q}{[k+p]_q - [p]_q + \lambda([k+p]_q^2 - [p]_q^2 + \alpha)},$$

then

$$N_{\delta,q}(g) \subseteq S_q(p,\alpha,\lambda).$$

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