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On additively regular property in $C(X)$

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Abstract. A commutative ring R is called *additively regular* if for each regular element $f \in R$ and each $g \in R$, there is an element $t \in R$ such that $g + ft$ is regular. In this note, we focus on $C(X)$ to study spaces X when $C(X)$ is an additively regular ring.

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1. Introduction

Through out this note, all rings are **commutative** with $1 \neq 0$, $\mathrm{U}(R)$ its group of units, $\mathrm{Idem}(R)$ its set of idempotents, The annihilator of a subset X of a ring R is denoted by $\mathrm{Ann}(X)$ and $\mathrm{Ann}(X) = \{r \in R | rX = 0\}$. When $\mathrm{Ann}(x) \neq 0$ we say x is a zero-divisor; otherwise it is a regular. The set of zero-divisor elements (resp. regular elements) of R is denoted by $\mathrm{Zdv}(R)$ (resp. $\mathrm{reg}(R)$). Let $Q_{cl}(R) = \left\{\frac{a}{r} : a, r \in R \,\text{and}\, r \,\text{is regular}\right\}$ denote the classical ring of fractions (total ring of quotients) of R.

Recall that for a ring R, R is *additively regular* if for each regular element $f \in R$ and each $g \in R$, there is an element $t \in R$ such that $g+ft$ is regular. (Note that if g is regular, we can simply choose $t = 0$.) Additively regular rings were implicit in [9], and were named by Gilmer and Huckaba in [3]. It is easy to see that a ring R is additively regular if and only if for each $x \in Q_{cl}(R)$, there exists $r \in R$ such that $x + r$ is a regular element of $Q_{cl}(R)$. The aim of this note is to study the additively regular property in $C(X)$.

$C(X)$ is the ring of all continuous real-valued functions on a completely regular Hausdorff space X. For $f \in C(X)$ the zero-set of f is the set $Z(f) = \{x \in X : f(x) = 0\}$. The set-theoretic complement of $Z(f)$ is denoted by $coz(f)$ and is called the cozero-set of f. The Stone-Čech compactification of X is denoted by βX. We refer the reader to [4], for undefined terms and notations.

2. Main results

We begin with the following definition.

2.1. Definition. For a pair of elements g and f in a ring R, we say the ordered pair of elements (g, f) is an *additively regular pair* if there is an element $t \in R$ such that $g + ft$ is regular. We say a ring R is *additively regular* if for each regular element $f \in R$ and each $g \in R$, there is an element $t \in R$ such that $g + ft$ is regular.

A ring in which every regular element is a unit is called a *classical ring*, and various examples are provided, see [8, pp. 320-322] for more details. It is easy to see that every classical ring is additively regular. It is known that $C(X)$ is a classical ring if and only if X is an almost P-space, see [1, p.19]. Recall that a space X is said to be a *almost* P*-space* if every nonempty zero-set of X has nonempty interior.
Now we make the following result.

2.2. Corollary. *If X is an almost* P*-space, then $C(X)$ is an additively regular ring.*

Recall that a ring R is called *von Neumann regular* if for each $a \in R$ there exists $b \in R$ such that $a = a^2 b$. A ring R is called *complemented* if for every $a \in R$ there is a $b \in R$ such that $ab = 0$ and $a + b$ is a regular element of R. It is not hard to see that a complemented ring is reduced.

*speaker

It is well known that R is complemented if and only if $Q_{cl}(R)$ is von Neumann regular, see [7]. Although the following result is not new (see [5, Theorem 7.4]), we shall include its proof for the sake of completeness.

2.3. Lemma. *Let R be a ring. If $Q_{cl}(R)$ is von Neumann regular then R is an additively regular ring.*

Proof. Let $f \in \mathrm{reg}(R)$ and $g \in \mathrm{Zdv}(R)$. By hypothesis, there is a $t \in R$ such that $gt = 0$ and $g + t$ is a regular. It suffices to show that $g + tf$ is a regular. Suppose that there exists $x \in R$ such that $x(g + tf) = 0$. Thus, $xg = -xtf$ and so $xg^2 = -xtfg = 0$. Thus, we have $(xg)^2 = 0$. Since R is a reduced ring, we infer that $xg = 0$. Since $x(g + tf) = 0$, we infer that $xtf = 0$. This implies that $xt = 0$ for $f \in \mathrm{reg}(R)$. Hence, we have $x(g + t) = 0$ and this is a contradiction. Thus, $g + tf$ is a regular, as desired. □

A space X is said to be *cozero complemented* if for each $f \in C(X)$, there is a $g \in C(X)$ such that the union of their cozerosets is dense and intersection of their cozerosets is empty. For examples every perfectly normal space and every metrizable space are cozero complemented. It is known that X is cozero complemented if and only if $Q_{cl}(X)$ is a von Neumann regular ring, for example see [6, Theorem 1.3]. Using Lemma 2.3, we make the following.

2.4. Corollary. *If X is a cozero complemented, then $C(X)$ is an additively regular ring.*

Before giving the next result, we need some lemmas.

2.5. Lemma. *Let $g \in C(X)$ and $f \in \mathrm{reg}(C(X))$. If g does not change sign, then (g, f) is an additively regular pair.*

Proof. **Case 1.** Suppose that $g \geq 0$. Take $t = f$. Hence, we have $Z^\circ(g + ft) = Z^\circ(g + f^2) \subseteq Z^\circ(f^2) = Z^\circ(f) = \emptyset$.
Case 2. Suppose that $g \leq 0$. Take $t = -f$. The proof is similar to case (1). □

A subspace S of X is called *C^*-embedded in X* if every function in $C^*(S)$ can be extended to a function in $C^*(X)$.

2.6. Lemma. *Let $g \in C(X)$ and $f \in \mathrm{reg}(C(X))$. If $coz(g)$ is C^*-embedded in X, then (g, f) is an additively regular pair.*

Proof. It is clear that $sign(g)$ is a bounded continuous function on $coz(g)$. By hypothesis, there exists $k \in C(X)$ such that $k|_{coz(g)} = sign(g)$. It is obvious that k is unit in $C(X)$ and $|g| = kg$ does not change sign. Using Lemma 2.5, there is an $h \in C(X)$ such that $Z^\circ(|g| + hf) = \emptyset$. Taking $t = k^{-1}h$, we have

$$\emptyset = Z^\circ(|g| + fh) = Z^\circ((kg + fkt)) = Z^\circ(k(g + ft)) = Z^\circ(g + ft).$$

□

A ring R is called *Bézout* if every finitely generated ideal is principal. A space X is an *F-space* if its every cozero-set is C^*-embedded. Examples of F-spaces are basically disconnected spaces (e.g., P-spaces and extremally disconnected spaces). It is known that $C(X)$ is a Bézout ring if and only if X is an F-space, see [4, 14.25].
With the help of Lemma 2.6, we have the following observation.

2.7. Corollary. *If X is an F-space, then $C(X)$ is an additively regular ring.*

For the proof of the next result, we need the following Lemma. For simplicity sake for any $g, f \in C(X)$, we will set $A_r := Z(g + rf)$ for each $r \in \mathbb{R}$.

2.8. Lemma. *Let $f, g \in C(X)$ and $f \in \mathrm{reg}(C(X))$. Then $A_r^\circ \cap A_s^\circ = \emptyset$ for any two distinct real numbers r and s.*

PROOF. Assume that $x \in A_r \cap A_s$. Hence, we have $g(x) + rf(x) = 0 = g(x) + sf(x)$. This yields $rf(x) = sf(x)$ and so $f(x) = 0$. Since $g(x)+rf(x) = 0$ and $f(x) = 0$, we infer that $g(x) = 0$. Hence, we have $x \in Z(g) \cap Z(f)$, a contradiction. $\square$

The smallest cardinal α such that every family of pairwise disjoint nonempty open subsets of X has cardinality less than or equal to α is called the *Souslin number* or the *cellularity* of X.

2.9. COROLLARY. *If the Souslin number of X is less than $\aleph_1$, then $C(X)$ is an additively regular ring.*

PROOF. Assume that $g, f \in C(X)$ and $f \in \mathrm{reg}(C(X))$. Using Lemma 2.8, we infer that $\{A_r^\circ\}_{r\in\mathbb{R}}$ is a family of pairwise disjoint open subsets of X. Since the Suslin number of X is less than $\aleph_1$, there exists $r' \in R$ such that $Z^\circ(g + fr') = \emptyset$. This yeilds $g + fr'$ is a regular, as desired. $\square$

2.10. LEMMA. *Let $g \in C(X)$ and $f \in \mathrm{reg}(C(X))$. If $\mathrm{Im}(\frac{g}{f}) \neq \mathbb{R}$, then (g, f) is an additively regular pair.*

PROOF. Let $r \notin \mathrm{Im}(\frac{g}{f})$. Clearly, $g(x)-rf(x) \neq 0$ for every $x \in coz(f)$. Therefore, $Z^\circ(g-rf) \subseteq Z^\circ(f) = \emptyset$. $\square$

A subspace S of X is called *C-embedded* in X if every function in $C(S)$ can be extended to a function in $C(X)$.

2.11. LEMMA. *Let $g \in C(X)$ and $f \in \mathrm{reg}(C(X))$. If $coz(f)$ is C-embedded in X, then (g, f) is an additively regular pair.*

PROOF. Clearly, $\frac{g}{f} + 1$ is a continuous function on $coz(f)$ and so it has an extension $t \in C(X)$. Hence, we have

$$Z^\circ(g + ft) \subseteq Z^\circ(f) = \emptyset.$$

$\square$

The following is an easy consequence of Lemma 2.11.

2.12. COROLLARY. *Let $g \in C(X)$ and $f \in \mathrm{reg}(C(X))$. If f is a unit in X, then (g, f) is an additively regular pair.*

2.13. LEMMA. *Let $f, g \in C(X)$ and $f \in \mathrm{reg}(C(X)$. If $Z(f) \subseteq Z^o(g)$ then (g, f) is an additively regular pair.*

PROOF. Assume that $f, g \in C(X)$ and f is a regular element of $C(X)$. Without loss of generality, we may assume that g is not regular. Using [4, 1D], g is a multiple of f, say $g = fh$ where $h \in C(X)$. Hence $g + f(1 - h)$ is a regular, as desired. $\square$

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