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Modular Riesz Bases and Woven Modular g-frame in Hilbert C^*-modules

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Abstract. In this paper we introduce modular frame, woven modular frame in Hilbert C^*-modules. And we show that they share many properties with Riesz basis in Hilbert. Also, we study some properties of these operators. We have discussed the relations and properties between woven Riesz bases and Riesz bases in C^*. Finally, we have discussed the properties of woven g-frames for tensor product of Hilbert space.
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1. Introduction

Hilbert space frames were originally introduced by Duffin and Schaeffer to deal with some problems in non-harmonic Fourier analysis[5]. Frames can be viewed as redundant bases which are generalizations of Riesz bases [1–4]. This redundancy property sometimes is extremely important in applications such as signal and image processing, data compression and sampling theory. In recent years, many mathematicians got significant results by extending the theory of frames from Hilbert spaces to Hilbert C^*-Modules. Hilbert C^*-Modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of real or complex numbers. They were introduced and investigated initially by Kaplansky [1, 2]. Frank and Larson [6] introduced the concept of frames in finitely or countably generated Hilbert C^*-modules over a unital C^*-algebra. In [1, 2], A. Khosravi and B. Khosravi introduced g-frames in Hilbert C^*-modules and observed that they share many useful properties with their corresponding notions in Hilbert spaces. Frames for Hilbert spaces have natural analogues for Hilbert C^*-modules. These frames are called Hilbert C^*-modular frames or just simply modular frames. Modular frames are not trivial generalizations of Hilbert space frames due to the complex structure of C^*-algebras. It is well known that the theory of Hilbert C^*-modules is quite different from that of Hilbert spaces. For example, we know that, any closed linear subspace in a Hilbert space has an orthogonal complement. But this is no longer true in Hilbert C^*-module setting since not every closed submodule of a Hilbert C^*-module is complemented. Moreover, the Riesz representation theorem for continuous functionals on Hilbert spaces does not hold in Hilbert C^*-modules, and so there exist nonadjointable bounded linear operators on Hilbert C^*-modules [1, 2]. Therefore it is expected that problems about frames in HilbertC^*-modules are more complicated than those in Hilbert spaces. While some of the results about frames in Hilbert spaces can be easily extended to Hilbert C^*-modular frames, many others cannot be obtained by simply modifying the approaches used in Hilbert spaces case.

2. Woven Modular g-frame

In this section, first we recall some definitions and basic properties of Hilbert C^*- Modules and p-woven frame and g-frame in Hilbert C^*- Modules. Throughout this note A is a unital C^*-algebra and H, K_i are finitely or countably generated Hilbert A-modules. For each $i \in I$, $L(H, K_i)$ will denote the set of all adjointable A-linear maps from H to K_i. We also define

$$\ell^2(A) := \{a = (a_i) \in A : \sum_{i\in I} a_i^* a_i \text{ is norm convergent in } A\}$$

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2.1. Definition. A pre-Hilbert A-module is a left A-module H equipped with an A-valued inner product $\langle .,. \rangle : H \times H \longrightarrow A$, such that
(i) $\langle x,x\rangle \geq 0$ for all $x \in H$ and $\langle x,x\rangle = 0$ if and only if $x = 0$,
$(ii)\langle x,y\rangle = \langle y,x\rangle^*$ for all $x, y \in H$,
(iii) $\langle ax+y,z\rangle = a\langle x,z\rangle + \langle y,z\rangle$ for all $a \in A$ and $x, y, z \in H$.

We assume that the linear operations of A and H are compatible.i.e. $\lambda(ax) = (\lambda a)x$ for every $\lambda \in \mathbb{C}$, $a \in A$ and $x \in H$. For every $x \in H$, we define

$$\| x \| = \| \langle x,x\rangle \|^{\frac{1}{2}} \qquad \text{and} \qquad |x| = \langle x,x\rangle^{\frac{1}{2}}.$$

If the pre-Hilbert A-module $(H, \langle .,. \rangle)$ is complete with respect to $\| \ . \ \|$, it is called a Hilbert A-module or a Hilbert C^*-modules over A. In this paper we focus on finitely and countably generated Hilbert C^*-modules over unital C^*-algebra A. A Hilbert A-module H is (algebraically) finitely generated if there exists a finite subset $\{x_1, x_2, ..., x_m\}$ of H such that every element $x \in H$ can be expressed as an A-linear combination $x = \sum_{i=1}^{m} a_i x_i, a_i \in A$. A Hilbert A-module H is countably generated if there exists a countable set of generators.
We now recall the definitions of frames and Riesz bases in Hilbert C^*-modules as follows.

2.2. Definition. Let H be a Hilbert A-module. A family $\{x_i : i \in I\}$ of elements of H is a (standard) frame for H, if there exits constants $0 < C \leq D < \infty$, such that for all $x \in H$,

$$C\langle x,x\rangle \leq \sum_{i\in I}\langle x,x_i\rangle\langle x_i,x\rangle \leq D\langle x,x\rangle. \qquad (1)$$

Where the sum in the middle of the inequality convergent in norm for $x \in H$.
The numbers C and D are called frame bounds, If $C = D = \lambda$, it is called a λ-tight frame and when $C = D = 1$, it is called a Parseval frame. $\{x_i : i \in I\}$ is said to be a Bessel sequence if only the right-hand side inequality is required.If the sum of (1) is convergent in norm, the frame is called standard.

According to what Arambasic and Khosravi proved, the above definition is equivalent to,

$$C \| x \|^2 \leq \| \sum_{i\in I}\langle x,x_i\rangle\langle x_i,x\rangle \| \leq D \| x \|^2,$$

A sequence $\{x_i : i \in I\}$ is said to be a Riesz basis of H if it is a frame and a generating set with the additional property that A-linear combinations $\sum_{i\in S} a_i x_i$ with coefficients $\{a_i : i \in S\} \subseteq A$ and $S \subseteq I$ are equal to zero if and only if in particular every summand $a_i x_i$ equal zero for $i \in S$. Note that we can also define the analysis operator, synthesis operator and frame operator for modular frame as follows. Suppose that $\{x_i : i \in I\}$ is a frame of a finitely or countably generated Hilbert A-module H over a unital C^*-algebra A. The operator $T : H \to \ell^2(A)$ defined by $Tx = \{\langle x,x_i\rangle\}_{i\in I}$, is called the analysis operator. The adjoint operator $T^* : \ell^2(A) \to H$ is given by $T^*\{a_i\}_{i\in I} = \sum_{i\in I} a_i x_i$. T^* is called pre-frame operator or the synthesis operator. By composing T and T^*, we obtain the frame operator $S : H \to H$,

$$Sx = T^*Tx = \sum_{i\in I}\langle x,x_i\rangle x_i, \qquad (1)$$

is a frame operator for H. That is $S \in End_A^*(H)$, positive and invertible. Where $End_A^*(H)$ is the set of adjointable A-linear maps on H.
The frame $\{S^{-1}x_i : i \in I\}$ is said to be the canonical dual frame of $\{x_i : i \in I\}$.

2.3. Definition. Let H be a Hilbert A-module. We say that $\{x_i^j : i \in I\}$ for $j = 1, 2, ..., m$ is a p-woven frame, if sequence $\{x_i^j : i \in I\}$ for $j = 1, 2, ..., m$ be Bessel and there exists a partition $p = \{\sigma_1, \sigma_2, ..., \sigma_m\}$ of I such that $\bigcup_{j=1}^{m}\{x_i^j : i \in \sigma_j\}$ is a frame.

2.4. Definition. Let H be a Hilbert C^*-module over a unital C^*-algebra A. Suppose $\{x_i^j : i \in I\}$ for $j = 1, 2, ..., m$ is a Bessel sequence in H. we say that $\{x_i^j : i \in I\}$ for $j = 1, 2, ..., m$ is a p-woven modular Riesz basis for H if there exists a partition to $p = \{\sigma_1, \sigma_2, ..., \sigma_m\}$ of I such that $\bigcup_{j=1}^{m}\{x_i^j : i \in \sigma_j\}$ is a modular Riesz basis for H.

Since $\{x_i^j : i \in \sigma_j\}$ for $j = 1, 2, ..., m$ is a Bessel sequence, we can define the synthesis operator T_j, the analysis operator T_j^* and the frame operator S_j, if $\{x_i^j : j = 1, 2, ..., m, i \in \sigma_j\}$ is a p-woven Riesz and a partition $p=\{\sigma_1,\sigma_2,...,\sigma_m\}$ of I, then $S_p(x)=\sum_{j=1}^m S^j_{\sigma_j}(x)$ where $S^j_{\sigma_j}(x)=\sum_{i\in\sigma_j}\langle x,x_i^j\rangle x_i^j$ for every j, $S^j_{\sigma_j}$ is a bounded, self adjoint and positive operator.

2.5. THEOREM. *Let H be a finitely or countably Hilbert A-module. Then $\{x_i^j : i\in I, j=1,2,...,m\}$ is a p-woven modular Riesz basis in H if and only if the following hold:*
$(i)\bigcup_{j=1}^m\{x_i^j : i\in\sigma_j\}$ is a Riesz basis.
$(ii)\{x_i^j : i\in I\}$ for $j=1,2,...,m$ has a unique dual frame $\{y_i^j : i\in I\}$ for $j=1,2,...,m$ which is a p-woven modular Riesz basis.

PROOF. (i) Let $\{x_i^j : i\in I, j=1,2,...,m\}$ be a modular Riesz basis in H corresponding to partition $p=\{\sigma_1,\sigma_2,...,\sigma_m\}$ of I. Then $\bigcup_{j=1}^m\{x_i^j : i\in\sigma_j\}$ is a frame for H and if $\sum_{i\in\sigma_j}a_i^jx_i^j=0$, then $a_i^jx_i^j=0$ for each $i\in\sigma_j, j=1,2,...,m$. Hence (i) is proved.
(ii) Since $U^j\in L(\ell^2(A),H)$ is invertible and $x_i^j=U^j(e_i^j)$ for each $i\in I$, then for every $x\in H$, $(U^j)^{-1}(x)\in\ell^2(A)$, defined $U=\oplus_{j=1}^m U^j$ we have

$$U^{-1}(x)=\sum_{j=1}^m\sum_{i\in\sigma_j}\langle(U^j)^{-1}(x),e_i^j\rangle e_i^j=\sum_{j=1}^m\sum_{i\in\sigma_j}\langle x,((U^j)^{-1})^*e_i^j\rangle e_i^j.$$

Therefore $x=U(U^{-1}x)=\sum_{j=1}^m\sum_{i\in\sigma_j}\langle x,((U^j)^{-1})^*e_i^j\rangle e_i^j$.
Now $(U^{-1})^*=\oplus_{j=1}^m((U^j)^{-1})^* : \ell^2(A)\to H$ is adjointable and invertable. It follows that $\{y_i^j=((U^j)^{-1})^*e_i^j : i\in I\}$ is a modular Riesz basis for H and for every

$$x\in H, x=\sum_{j=1}^m\sum_{i\in\sigma_j}\langle x,y_i^j\rangle x_i^j.$$

Therefore $\{y_i^j : i\in I\}$ for $j=1,2,...,m$, is a dual p-woven of $\{x_i^j : i\in I\}$ for $j=1,2,...,m$ and by (i) it is unique. □

2.6. REMARK. Let H and K be Hilbert spaces. Then the tensor product of H and K is the set $H\otimes K$ of all antilinear maps $T : K\to H$ such that $\sum_i\|\ Tu_i\ \|^2<\infty$ for some , and hence every orthonormal basis $\{u_i : i\in I\}$ of K .Moreover for every $T\in H\otimes K$ we set

$$|||\ T\ |||^2=\sum_i\|\ Tu_i\ \|^2$$

By Theorem $7-12$ in [7] . $H\otimes K$ is a Hilbert space with the norm $|||\ .\ |||$ and associated inner product $<T,\Lambda>=\sum_i<Tu_i,\Lambda u_i>$ where $\{u_i : i\in I\}$ is an arbitrary orthonormal basis of K . Let $x\in H$ and $y\in K$ .Then we define the map $x\otimes y$ by

$$(x\otimes y)(z)=<y,z>x\ (z\in K)$$

Obviously $x\otimes y$ belongs to $H\otimes K$, Let $T\in H\otimes K$. $x,x'\in H$ and $y,y'\in K$, then by Theorem 7.12 in [7]

$$|||\ T\ |||=|||\ T^*\ |||\ ,\ |||\ x\otimes y\ |||=\|\ x\ \|\|\ y\ \|\ ,\ <x\otimes y,x'\otimes y'>=<x,x'><y,y'>$$

If $\{e_i : i\in I\}$ and $\{u_i : j\in\}$ are orthonormal bases for H and K respectively, then $\{e_i\otimes u_j : i\in I, j\in J\}$ is an orthonormal basis for $H\otimes K$. By Theorem 7.12 in [7].

2.7. THEOREM. *Let $\{f_i^j : i\in I\}$ for $j=1,2,...,m$ be a p-woven frame for H and $\{g_l^k : l\in J\}$ for $k=1,2,...,n$ be a p-woven frame for K . Then $\{f_i^j\otimes g_l^k : (i,j)\in I\times J\}$ for $j=1,2,...,m$ and $k=1,2,...,n$ is a p-woven frame for $H\otimes K$*

PROOF. Let $\bigcup_{j=1}^m\{f_i^j : i\in\sigma_j\}$ is a frame for H corresponding to $P=\{\sigma_1,\sigma_2,...,\sigma_m\}$ and $\bigcup_{k=1}^n\{g_l^k : l\in\delta_k\}$ is a frame for K corresponding to $P'=\{\delta_1,\delta_2,...,\delta_n\}$. Then by Theorem 2-2 in [7], $\bigcup_{j=1}^m\bigcup_{k=1}^n\{f_i^j\otimes g_l^k : i\in\sigma_j, l\in\delta_j\}$ is a frame for $H\otimes K$ corresponding to the partition $P''\{\Delta_{j,k}=\sigma_j\times\delta_k : j=1,2,...,m, k=1,2,...,n\}$ of $I\times J$. □

2.8. DEFINITION. A sequence $\{\Lambda_i\in L(H,K_i) : i\in I\}$ is a g-frame in Hilbert $A-$module H with respect to $\{K_i : i\in I\}$, if there exist real constants C,D such that for every $x\in H$

$$C\langle x,x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq D\langle x,x\rangle. \qquad (2)$$

We call C and D the g-frame bounds. If only the right-hand sid is required, it is called a g-Bessel sequence. Moreover if $C = D = \lambda$ tight and if $C = D = 1$, it is called g-Parseval. The g-frame is standard if for each $x \in H$, the sum in (2) converges in norm.

2.9. Theorem. *Let $\{\Lambda_i^j \in L(H_j, K_i^j) : i \in I\}$ for $j = 1, 2, ..., m$ be a g-frame with bounds A_i, B_i. Then $\{\Lambda_i^1 \otimes \Lambda_i^2 \otimes ... \otimes \Lambda_i^m : i \in I\}$ is a g-frame for $H_1 \otimes ... \otimes H_m$ w.r.t $\{K_i^1 \otimes K_i^2 \otimes ... \otimes K_i^m : i \in I\}$*

Proof. By the associativily of tensor product [6, *Prop.*2.6.5] it is enough to prove the theorem for $n = 2$. Let $\{\Lambda_i \in L(H_1, K_i) : i \in I\}$ and $\{\Gamma_j \in L(H_2, M_j) : j \in J\}$ be two g-frames with bounds A_1, B_1 and A_2, B_2 respectively. For each $f \otimes g \in H_1 \otimes H_2$

$$\sum_{(i,j)\in I\times J} \langle (\Lambda_i \otimes \Gamma_j)(f\otimes g), (\Lambda_i \otimes \Gamma_j)(f\otimes g)\rangle$$

$$\sum_{(i,j)\in I\times J} \langle \Lambda_i(f)\otimes \Gamma_j(g), \Lambda_i(f)\otimes\Gamma_j(g)\rangle \leq B_1B_2\langle f,f\rangle\langle g,g\rangle$$

$$= B_1B_2\langle f\otimes g, f\otimes g\rangle.$$

Similarly, for lower bound

$$\sum_{(i,j)\in I\times J} \langle (\Lambda_i \otimes \Gamma_j)(f\otimes g), (\Lambda_i \otimes \Gamma_j)(f\otimes g)\rangle \geq A_1A_2\langle f\otimes g, f\otimes g\rangle.$$

Therefore we have the result. $\square$

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References

[1] A. Khosravi, B. Khosravi, g-Frames and modular riesz bases in Hilbert C^*- Modules, *J. Wavelets, Multiresoluion and Information Processing* **1o . No. 2** (2012).

[2] A. Khosravi, J. Sohrabi, *Weaving g-frames and weaving fusion frame*, *Bull. Malays. Math. Sci. Soc.* **42** (2019), 3111-3129.

[3] O. Christensen,*An Introduction to Frames and Riesz Bases*, 2nd edn. *Birkhauser, Boston*(2016).

[4] T. Bemrose, P.G. Casazza, K. Grochenig, M.C. Lammers, R.G. Lynch, *Weaving Frames Operators and Matrices*, Operators and Matrices**10** (2015), 110-114.

[5] R. J, Duffin, A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Am. Math. Soc. **72** (1952,) 341-366.

[6] M. Frank and D. R. Larson, *Frames in Hilbert C*-modules and C*-algebras*, *J. Operator Theory* **48** (2002), 273–314.

[7] G. B. Folland, *A Course in abstract harmonic analysis*, *CRC Press BOCA Raton* Florida. (1995).

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