Shahid Chamran University of Ahvaz

THE FIRST INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS

11-12 Augueest 2021, Shahid Chamran University of Ahvaz, Ahvaz, Iran

On Automorphic Classes of finite groups

S. Hosseini *

Department of Mathematics, Birjand Baranch, Islamic Azad University, Birjand, Iran.

Abstract. Let G be a finite group and $x \in G$. $Cl_A(x) = \{\alpha(x)|\alpha \in Aut(G)\}$ denotes the automorphic class of x in G. In this paper some properties of finite groups are described via automorphic class and it is shown that if the number of automorphic classes of a given lower autonilpotent group G, is even then $|Aut(G)|$ is also even.

Keywords: equivalence relation, absolute center, automorphism group..

AMS Mathematical Subject Classification [2010]: 20D45, 20D20,37A20.

1. Introduction

Let G be a finite group and A be its automorphism group. Additionally, let $L(G) = \{g \in G|\alpha(g) = g, \forall \alpha \in Aut(G)\}$ and $K(G) = \langle [g, \alpha] = g^{-1}g^{\alpha}|g \in G, \alpha \in Aut(G)\rangle$ be the absolute center and autocommutator subgroup of G respectively. The concepts of absolute center and autocommutator has been introduced by Hegarty in [2]. The autcommutator of higher weight is defined inductively as follows:
$[g, \alpha_1, \alpha_2, ..., \alpha_i] = [[g, \alpha_1, \alpha_2, ..., \alpha_{i-1}], \alpha_i],$ *for all* $\alpha_1, \alpha_2, ..., \alpha_i \in Aut(G), i \geq 2$. The autcommutator subgroup of weight i is defined as:
$K_i(G) = [G, \underbrace{Aut(G), ..., Aut(G)}_{i \text{ times}}] = \langle [g, \alpha_1, \alpha_2, ..., \alpha_i]|g \in G, \alpha_1, \alpha_2, ..., \alpha_i \in Aut(G)\rangle$. Clearly $K_i(G)$ is a characteristic subgroup of G, for all $i \geq 1$. Thus we have a descending chain of autocommutator subgroups $G = K_0(G) \supseteq K_1(G) \supseteq K_2(G) \supseteq ... \supseteq K_i(G) \supseteq ...$, called the lower autocentral series of G. A group G is said to be lower autonilpotent of class n if $K_n(G) = 1$ and $K_{n-1}(G) \neq 1$. See more in [1].

1.1. Definition. Let G be a finite group. $H = St_A(x) = \{\alpha \in Aut(G)|\alpha(x) = x\}$ is called stabilizer of x in automorphism group of a group G.

Clearly, $H = St_A(x)$ is a subgroup of A. Since $Aut(G)$ acts as permutation group on G, define a relation $\sim$ on G by $x \sim y$ if there exists $\alpha \in Aut(G)$ with $\alpha(x) = y$, for $x, y \in G$. It is clear that $\sim$ is an equivalence relation and that G is partitioned into automorphism orbits. Let $Cl_A(x)$ be the orbit containing x which is called automorphic class of x in G. It is well-known that $|Cl_A(x)| = [Aut(G) : St_A(x)]$. Obviously if $Aut(G)$ is limited to $Inn(G)$, then $L(G) = Z(G)$ and automorphic class of G equals to conjugacy class of G. In the second section of this paper introductory properties of automorphic class of a given finite group are studied. In the final section, It is proved that even number of automorphic classes results in $Aut(G)$ of even order.

2. Some Properties of Automorphic Classes of finite groups

In the following example, we specify all automorphic classes of S_3.

2.1. Example. Let $S_3 \cong D_6 = \langle a, b|a^2 = b^3 = (ab)^2 = 1\rangle$.
$|Aut(S_3)| = 6$. Automorphisms of S_3 are

$\alpha_1 : \ a \mapsto a, \ b \mapsto b$ $\quad$ $\alpha_2 : \ a \mapsto ab, \ b \mapsto b$ $\quad$ $\alpha_3 : \ a \mapsto ab^2, \ b \mapsto b$

$\alpha_4 : \ a \mapsto a, \ b \mapsto b^2$ $\quad$ $\alpha_5 : \ a \mapsto ab, \ b \mapsto b^2$ $\quad$ $\alpha_6 : \ a \mapsto ab^2, \ b \mapsto b^2$.

Hence S_3 has 3 automorphic classes as follows:
$Cl_A(1) = \{1\}$, $Cl_A(a) = \{a, ab, a^2\}$, $Cl_A(b) = \{b, b^2\}$

2.2. Lemma. *Let* $Cl_A(x)$ *be an automorphic class of* G*. Then* $\langle Cl_A(x)\rangle$ *is characteristic.*

*speaker

2.3. Lemma. *Let H be a characteristic subgroup of finite group G and $|H| = p$ such that p is the least prime number that divides $|G|$, then $H \subseteq L(G)$.*

2.4. Theorem. *If $|G| = |Aut(G)|$ and $|Cl_A(x)| = 2$ for some $x \in G$, then $Aut(G)$ has a non trivial normal subgroup.*

3. Relation Between Automorphic classes Cardinality and $|Aut(G)|$

In this section we study relation between automorphic classes cardinality and the order of automorphism group of a finite group G. Furthermore, we prove that for lower autonilpotent groups, even automorphic classes cardinality results in automorphism group of even order.

3.1. Theorem. *Let G be a finite group and x is an element in G. Then there is a one to one correspondence between elements of $Cl_A(x)$ and right cosets of $St_A(x)$ in $Aut(G)$.*

3.2. Theorem. *Let G be a finite group then $|G| = L(G) + \sum_{i=1}^{r}[Aut(G) : St_A(g_i)]$ where $g_1, ..., g_r \in G - L(G)$.*

3.3. Theorem. *Let G be a finite group then $k^*(G)$, the number of automorphic classes of G is $k^*(G) = \frac{1}{Aut(G)} \sum_{x \in G} |St_A(x)|$.*

An authomorphism α of a group G is said to be Inverse Point Free (IPF) if $\alpha(x) = x^{-1} \Rightarrow x = 1 \ \forall x \in G$. Accordingly, G is called IPF if every $\alpha \in Aut(G)$ is IPF.

3.4. Lemma. *([4], Theorem 1.) G is IPF if and only if $| G || Aut(G) |$ is an odd number.*

3.5. Theorem. *Let G be a lower autonilpotent group. If the number of automorphic classes of G is even, then $|Aut(G)|$ is also even.*

Proof. If $Aut(G)$ is odd, then G is even and hence $| G || Aut(G) |$ is also even. Therefore by Lemma 3.4 G is not IPF which is a contradiction with lower autonilpotency of G and the proof is completed. □

References

[1] H. Arora and R. Karan, *On autonilpotent and autosoluble groups*, Note di Matematica, **38** (2018), 35-45
[2] P. V. Hegarty, *The Absolute Center of a Groups*, Journal of Algebra, **169** (1994), 929-935.
[3] M. R. R. Moghaddam, *Some Properties of Autocommutator Groups*, Presented at group theory conference, University of Isfahan, (March-2009), 12-13.
[4] F. Parvaneh, *Some Properties of autonilpotent Groups*, Italian Journal of Pure and Applied Mathematics, **35** (2015), 1-8.
[5] D. J. S. Robinson, *A Course in the Theory of Groups*, 2^{nd} ed. Springer-Verlag. New York, Berlin, 1996.

E-mail: sh.hoseini@iaubir.ac.ir