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Hyers-Ulam stabilities for harmonically s-convex function

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Abstract. In this paper we generalize the concept of harmonically convex to harmonically s-convex that s is a real number in $(\circ, 1]$ and attention to Sandwich theorem. Also, we investigate stability of Hyers-Ulam for them.

Keywords: Harmonically convex functions, Sandwich theorem, Hyers-Ulam stability.
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1 Introduction and Preliminaries

The notion of stability was expressed by Ulam in 1940 like this "what condition does there exist a homomorphism near an approximate homomorphism?". In 1941, Hyers answered to Ulam's problem on Banach spaces, in which case it's notion was called stability of Hyers-Ulam. Yet this notoin was browsed on operators and various spaces that first time it was expressed by Hyers-Ulam on convex function in 1952. Then the notion of ε-convex function and its stability was introduced and proved.
Convexity of functions and sets have important role in mathematical economics, engineering, management science, and optimization theory, (see [1, 3, 5]) thereby introducing new notions such as being quasi-convex, pseudo-convex, strongly convex, approximately convex, midconvex, h-convex, etc (see[4]). The classical sense of convex function has been extended and generalized in different directions. In this paper, we use a recent notion of generalized convexity, introduced by Isean [5] for s-convex function.
Definition 1.1. [5] Let I be an interval in $\mathbb{R}\backslash\{\circ\}$. A function $f : I \to \mathbb{R}$ is said to be harmonically convex on I if the inequality

$$f(\frac{xy}{tx+(1-t)y}) \leq tf(y)+(1-t)f(x), \tag{1.1}$$

[0] *Speaker

holds, for all $x, y \in I$ and $t \in [\circ, 1]$.

Definition 1.2. [2] Let s be a real number, $s \in (\circ, 1]$. A function $f : [\circ, \infty) \to [\circ, \infty)$ is said to be s-convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in [\circ, \infty)$ and $t \in [\circ, 1]$.

Motivated by the works mentioned above, we have the following definition.

Definition 1.3. Let $f : I \to \mathbb{R}$ and s be a real number in $(\circ, 1]$. We said that f is harmonically s-convex on I if the inequality

$$f(\frac{xy}{tx + (1-t)y}) \leq t^s f(y) + (1-t)^s f(x), \tag{1.2}$$

holds for all $x, y \in I$.

2 Main results

In this paper we have two main results. The first one is a sandwich theorem for harmonically s-convex functions and the second is about to we prove a stability result of Hyers-Ulam type for harmonically s-convex functions.

Proposition 2.1. Let $I \subseteq \mathbb{R} \backslash \{\circ\}$ be a real interval and $f : I \to \mathbb{R}$ a function. then:

- If $I \subseteq (\circ, +\infty)$ and f is s-convex and nondecreasing, then f is harmonically s-convex.
- If $I \subseteq (\circ, +\infty)$ and f is harmonically s-convex and nonincreasing, then f is s-convex.
- If $I \subseteq (-\infty, \circ)$ and f is harmonicaly s-convex and nondecreasing, then f is s-convex.
- If $I \subseteq (-\infty, \circ)$ and f is s-convex and nonincreasing, then f is harmonically s-convex.

Now, by using the proposition (2.1), we can obtain the sandwich theorem for harmonically s-convex functions.

Theorem 2.2. Let $f, g : I \to \mathbb{R}$ defined on a real interval I and $s \in (\circ, 1]$. Then

$$f(tx + (1-t)y) \leq t^s g(x) + (1-t)^s g(y)$$

for all $x, y \in I$ and $t \in [\circ, 1]$ if and only if there exists a s-convex function $k : I \to \mathbb{R}$ such that

$$f \leq k \leq g.$$

Remark 2.3. Theorem (2.2) generalized theorem 2 [3] to harmonically s-convex functions.

Theorem 2.4. Let f, g be real functions defined on the interval $(\circ, +\infty)$ and $s \in (\circ, 1]$. the following conditions are equivalent:

- There exists a harmonically s-convex function $k : (\circ, +\infty) \to \mathbb{R}$ such that $f(x) \leq k(x) \leq g(x)$, for all $x \in (\circ, +\infty)$.
- The following inequalitie holds:

$$f(\frac{xy}{tx + (1-t)y}) \leq t^s g(y) + (1-t)^s g(x),$$

for all $x, y \in (\circ, +\infty)$, $t \in [\circ, 1]$.

As an immediate consequence of above theorem, we obtain the following stability result of Hyers-Ulam type for harmonically s-convex functions.

Theorem 2.5. Let $[a, b] \subseteq (\circ, +\infty)$ be an interval, $\varepsilon > \circ$ and $s \in (\circ, 1]$. A function $f : [a, b] \to \mathbb{R}$ satisfies the inequality

$$|f(\frac{xy}{tx + (1-t)y}) - t^s f(y) - (1-t)^s f(x)| \leq \varepsilon,$$

for all $x, y \in [a, b]$ and $t \in [\circ, 1]$, if and only if there exists a harmonically s-convex function $\varphi : [a, b] \to \mathbb{R}$ such that

$$|f(x) - \varphi(x)| < \frac{\varepsilon}{2}, \quad \forall x \in [a, b].$$

References

[1] A. Barani, A. Ghazanfari and S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, Journal of Inequalities and Applications, **247** (2012), 247–256.

[2] K. Baron, J. Matkowski and K. Nikodem, *A sandwich with convexity.* Math. Pannica, **1** (1994), 139–144.

[3] M. Bessenyei and Zs. Páles, *Characterization of convexity via Hadamard's inequality*, Math. Inequal. Appl., **9** (2006), 53–62.

[4] A. Daniilidis and P. Georgiev, *Approximately convex functions and approximately monotonic operators*, Nonlin. Anal., **66** (2007), 547–567.

[5] I. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics, **6** (2014), 935–942.

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