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Cumulative Jensen Shannon information

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Abstract. Recently, a measure of information called Jensen-Shannon (JS) information was used to compare the coherent systems in reliability theory. JS is defined based on the notion of the vector of signature of coherent systems. The signature of a coherent system with n components is an n-dimensional vector whose ith element is the probability that the ith failure of the components is fatal to the system. JS information use to compare the coherent systems in terms of their complixity and ranks systems based on their designs. In this paper, we present the cumulative Jensen-Shannon (CJS) information as an alternative criteria to compare the coherent systems. Also, we show some properties of the mentioned measure and present an upper bound for CJS in terms of cumulative Kullback-Leibler information.
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1. Introduction

In the reliability theory literature, study the lifetime distribution of a system and its components based on notions of information theory has attracted the interest of many researchers. Measuring the uncertainty of lifetime distributions lead to predict the residual (past) lifetime of a system or its components as well as their failing times. A new stream of research applies the information theory of Shannon and Kullback-Leibler to compare systems in terms of the uncertainty and information about predicting the lifetimes [2, 5].

Asadi et al. (2016) proposed a useful link between of the information theory and the mixture representation of the system reliability function. They showed that the Jensen-Shannon divergence of the mixture distribution [21] provides information criteria for comparing systems solely based on the designs of the systems, without any assumption about the common distribution of the iid lifetimes of the components.

Consider a mixture probability density function (PDF)

$$f(y) = \sum_{i=1}^{n} s_i f_i(y), \quad s_i \geq 0, \quad \sum_{i=1}^{n} s_i = 1. \tag{1}$$

The Jensen-Shannon (JS) divergence of the mixture (1) is defined by

$$JS(f : f_1, \cdots, f_n;) = H(f) - \sum_{i=1}^{n} s_i H(f_i). \tag{2}$$

where $\mathbf{s} = (s_1, s_2, \cdots, s_n)$ and

$$H(f) = H(Y) = -\int_{-\infty}^{\infty} f(y) \log f(y) dy,$$

is the Shannon entropy of lifetime of the system. They showed

$$JS(f_T : f_{1:n}, \cdots, f_{n:n};) = JS(\mathbf{s}) = \sum_{i=1}^{n} s_i K(f_{i:n} : f_T), \tag{3}$$

where $\mathbf{s}$ is the vector of signature of the system and

$$K(f_{i:n} : f_T) = \int_0^{\infty} f_{i:n}(x) \log \frac{f_{i:n}(x)}{f_T(x)} dx$$

is the Kullback-Leibler divergence between the distributions of the ith order statistic and the system lifetime and $JS(\mathbf{s})$ is the weighted information discrepancy between the distributions of the order statistics and the distribution of the lifetime of the system. Also, they proposed a framwork to compare and sort coherent systems based on their signatures or the related complexity of each system, as well as some sharp bound for $JS(\mathbf{s})$. In next section, we defined a cumulative version of JS and present some new results in this connection.

2. Main results

Suppose a coherent system has n components whose lifetimes $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d) continuous random variables with distribution function F. (From another point of view random variables $X_1, \ldots, X_n$ can be the lifetime of heterogeneous components of a coherent system with distribution function $F_1, F_2, ..., F_n$, respectively). Let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the order statistics obtained by arranging the component lifetimes X_i's in increasing order of magnitude. Then the system lifetime T will coincide with an order statistic $X_{i:n}$ for some $i \in \{1, \cdots, n\}$, which let the following notion of *system signature* in a natural way. Let s_i, for $i = 1, \cdots, n$, be such that $P(T = X_{i:n}) = s_i$. Then, the system signature is the vector $\mathbf{s}=(s_1, \cdots, s_n)$.

Let T be the system lifetime and $\mathbf{s}$ its signature. Then,

$$\bar{F}_T(t) = P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t) \tag{4}$$

$$= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} [F(t)]^j [\bar{F}(t)]^{n-j} \tag{5}$$

Also for its density function we have

$$f_T(t) = \sum_{i=1}^{n} s_i f_{X_{i:n}}(t) \tag{6}$$

From representation (4), easily can see the expectation of T is as follows:

$$E(T) = \sum_{i=1}^{n} s_i E(X_{i:n}) \tag{7}$$

Note that one can consider the representation (4) , (6) as the mixture distribution of random variable T in terms of order statistics of an i.i.d random sample.

In the recent years many researchers have argued the concept of Cumulative Residual Entropy (CRE) as a measure of uncertainty. [see 1 , 4] The CRE which has a close relation with the shannon entropy of the equilibrium distributions is defined as

$$\varepsilon(F_T) = -\int \bar{F}_T(x) \log \bar{F}_T(x) dx \tag{8}$$

Also, based on CRE, Baratpour and Rad (2012) and other researchers have introduced the cumulative Kullback-Leibler information as a measure of discrimination between statistical models as follow

$$CRKL(F; G) = \int \bar{F}(x) \log \frac{\bar{F}(x)}{\bar{G}(x)} dx - (\mu_F - \mu_G). \tag{9}$$

where μ_F and μ_G are the finite expectation of the distributions F and G respectively [3, 5].

As the CRE is a concave function of $\bar{F}$ then from (4) we have

$$-\int \bar{F}_T(x) \log \bar{F}_T(x) dx \geq -\sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \bar{F}_{i:n}(x) dx \tag{10}$$

That is

$$\varepsilon(F_T) \quad \geq \quad -\sum_{i=1}^{n} s_i \varepsilon(F_{i:n}) \tag{11}$$

and equality holds if T be the lifetime of a coherent system.

2.1. Definition. The Cumulative Jensen-Shannon information $CJS(\mathbf{s})$ as a measure of divergence is defined as follows

$$CJS(\mathbf{s}) \quad = \quad \varepsilon(F_T) - \sum_{i=1}^{n} s_i \varepsilon(F_{i:n}) \geq 0 \tag{12}$$

2.2. Theorem. *The information measure $CJS(\mathbf{s})$ has the following representation in terms of CKL divergence between F_T and $F_{i:n}$.*

$$CJS(\mathbf{s}) \quad = \quad \sum_{i=1}^{n} s_i CKL(F_{i:n}; F_T) \tag{13}$$

proof: It is enough to show that

$$\sum_{i=1}^{n} s_i CKL(F_{i:n}; F_T) \quad = \quad \sum_{i=1}^{n} s_i [\int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}(x)}{\bar{F}_T(x)} dx - (\mu_{F_{i:n}} - \mu_{F_T})] \tag{14}$$

$$= \quad \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \bar{F}_{i:n}(x) dx - \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \bar{F}_T(x) dx \tag{15}$$

$$- \quad \sum_{i=1}^{n} s_i (\mu_{F_{i:n}} - \mu_{F_T}) \tag{16}$$

$$= \quad -\sum_{i=1}^{n} s_i \varepsilon(F_{i:n}) - \int (\sum_{i=1}^{n} s_i \bar{F}_{i:n}(x)) \log \bar{F}_T(x) dx - 0 \tag{17}$$

$$= \quad \varepsilon(F_T) - \sum_{i=1}^{n} s_i \varepsilon(F_{i:n}) \tag{18}$$

In following theorem we give an upper bound for CJS based on CKL.

2.3. Theorem. *$CJS(\mathbf{s})$ has the following upper bound in terms of CKL divergence between $F_{i:n}$'s.*

$$CJS(\mathbf{s}) \quad \leq \quad \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j CKL(F_{i:n}; F_{j:n}) \tag{19}$$

proof: We have

$$
\begin{aligned}
CJS(\mathbf{s}) &= \sum_{i=1}^{n} s_i CKL(F_{i:n}; F_T) && (20)\\
&= \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}(x)}{\bar{F}_T(x)} dx && (21)\\
&= \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}(x)}{\sum_{i=1}^{n} s_i \bar{F}_{i:n}(x)} dx && (22)\\
&\leq \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}(x)}{\prod_{i=1}^{n} \bar{F}_{i:n}^{s_i}(x)} dx && (23)\\
&= \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}^{s_1}(x)}{\bar{F}_{1:n}^{s_1}(x)} dx + \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}^{s_2}(x)}{\bar{F}_{2:n}^{s_2}(x)} dx + && (24)\\
&+ \ldots + \sum_{i=1}^{n} s_i \int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}^{s_n}(x)}{\bar{F}_{n:n}^{s_n}(x)} dx && (25)\\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \int \bar{F}_{i:n}(x) \log \frac{\bar{F}_{i:n}(x)}{\bar{F}_{j:n}(x)} dx && (26)\\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j CKL(F_{i:n}; F_{j:n}) && (27)
\end{aligned}
$$

At present, we are working on propeties and merits of CJS.

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