THE FIRST INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS
11-12 Augeust 2021, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Variational inequality over the intersection of fixed point sets of generalized demimetric mappings and zero point sets of maximal monotone mappings

Mohammad Eslamian *
Department of Mathematics, University of Science and Technology of Mazandaran

Abstract. In this paper, we consider a variational inequality problem which is defined over the intersection of the set of common fixed points of a finite family of generalized demimetric mappings and the set of common zero points of a finite family of maximal monotone mappings. We propose an iterative algorithm combines the hybrid steepest descent method with the inertial technique to solve such a variational inequality problem and study the strong convergence of the sequence generated by the proposed algorithm. The results of this paper improve and extend several known results in the literature.
Keywords: Variational inequality; maximal monotone; generalized demimetric mappings.
AMS Mathematical Subject Classification [2010]: 49J30, 47H09, 47J20.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$. Recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is called L–Lipschitzian if for all $x, y \in \mathcal{H}$, $\|Tx - Ty\| \le L\|x - y\|$, where $L > 0$ is a constant. In particular, if $L = 1$ then T is called a nonexpansive mapping. A mapping $T : \mathcal{H} \to \mathcal{H}$ is called strongly monotone with constant $\beta > 0$, if

$$\langle T(x) - T(y), x - y\rangle \ge \beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

We denote by $Fix(T)$ the set of fixed points of the mapping T; namely, $Fix(T) := \{x \in \mathcal{H} : Tx = x\}$.

Numerous problems in science and engineering, including optimization problems, fixed point problems, transportation problems, financial equilibrium problems, migration equilibrium problems etc. lead to study variational inequality problems (VIP) on a closed convex feasible set of a Hilbert space (see [1]). The variational inequality problem is to find $x \in C$ such that

$$\langle Fx, y - x\rangle, \qquad \forall y \in C.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, F)$. It is known that if F is a strongly monotone and Lipschitzian mapping on C, then $VI(C, F)$ has a unique solution. It is well-known that VIP on a closed convex feasible subset of a Hilbert space is equivalent to the fixed point equation containing the metric projection from any point in Hilbert space onto the feasible set. In some special cases, the metric projection can be expressed in closed form, then VIP can be solved easily by the gradient projection method. In general, it is difficult to compute the projection on any closed convex set. To overcome this disadvantage, which is caused by the metric projection, Yamada [2] introduced a new algorithm, namely, the hybrid steepest descent method, for solving VIP in which the feasible set is expressed as the fixed point set of a nonexpansive mapping. The method is defined by the following:

$$x_{n+1} = (I - \mu\alpha_n F)Tx_n,$$

where F is a Lipschitzian continuous and strongly monotone mapping and T is a nonexpansive mapping. Variational inequalities over the fixed point set of nonexpansive mappings have an important role in solving practical problems such as the signal recovery problem, beamforming

*speaker

problem, power control problem, bandwidth allocation problem and finance problem (see, e.g., [3]). The hybrid steepest-descent method has received great attention given by many authors, see e.g., [4, 5] and the references therein. In 2014, Zhou and Wang [4] considered the VIP with the feasible set C is the intersection of fixed point sets of a finite family of nonexpansive mappings on $\mathcal{H}$. They proposed an iterative algorithm and proved a strong convergence theorem based on Yamada's hybrid steepest descent method and Mann's iterative method.

1.1. Theorem. *Let $\mathcal{H}$ be a real Hilbert space and $F : \mathcal{H} \to \mathcal{H}$ be an L-Lipschitz continuous and η-strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be nonexpansive self mappings of $\mathcal{H}$ such that $C = \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. For arbitrary initial point $x_0 \in \mathcal{H}$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = (I - \mu\alpha_n F)T_N^n T_{N-1}^n ... T_1^n x_n \tag{1}$$

where $\mu \in (0, \frac{2\eta}{L^2})$ and $T_i^n = (1-\lambda_n^i)I + \lambda_n^i T_i$. Assume that $\{\lambda_n^i\}$ and $\{\alpha_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$;
(ii) $\{\lambda_n^i\} \subset (a,b)$ for some $a, b \in (0,1)$ for $i = 1, 2, ..., N$;
(iii) $\lim_{n\to\infty} |\lambda_{n+1}^i - \lambda_n^i| = 0$ for $i = 1, 2, ..., N$.

Then, the sequence $\{x_n\}$ converges strongly to the unique solution $x^\star \in VI(C, F)$.

In 2018, Kawasaki and Takahashi [6], introduced a new general class of mappings, called generalized demimetric mappings as follows:

1.2. Definition. Let ζ be a real number with $\zeta \neq 0$. A mapping $T : \mathcal{H} \to \mathcal{H}$ with $Fix(T) \neq \emptyset$ is called generalized demimetric, if

$$\zeta\langle x - x^\star, x - Tx\rangle \geq \|x - Tx\|^2,$$

for all $x \in \mathcal{H}$ and $x^\star \in Fix(T)$. This mapping T is called ζ-generalized demimetric.

We observe that the class of generalized demimetric mappings covers the classes of well-known mappings such as strict pseudo-contraction, quasi-nonexpansive and demicontractive mappings, see [6, 7] for details.

On the other hand, in 2001, Alvarez and Attouch [8] proposed an inertial proximal algorithm to find zero point of a maximal monotone operator:

$$x_{n+1} = J_{\lambda_n}^A (x_n + \alpha_n(x_n - x_{n-1})), \quad n \geq 1 \tag{2}$$

where the set-valued mapping $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator and the function $J_{\lambda_n}^A = (I + \lambda_n A)^{-1}$ is the resolvent of A. The extrapolation term $\alpha_n(x_n - x_{n-1})$ takes into account an inertial effect of this algorithm. Under some mild conditions, most iterative algorithms by using this inertial effect have better convergence behavior for various problems, such as the inclusion problem, the variational inequality problem, and the fixed point problem (see [9]).

In this paper, we consider a variational inequality problem which is defined over the intersections of the set of common fixed points of a finite family of generalized demimetric mappings and the set of common zero points of a finite family of maximal monotone mappings. We propose an iterative algorithm combines the steepest descent method with the inertial technique to solve such a variational inequality problem and study the strong convergence of the sequence generated by the proposed algorithm.

2. Preliminaries

In this section, we summarize some notations, definitions and lemmas which play significant role in convergence analysis of our algorithm. Throughout the paper, we adopt the following notations:

- $x_n \to x$ stands for the strong convergence $\{x_n\}$ to x.
- $x_n \rightharpoonup x$ stands for the weak convergence $\{x_n\}$ to x.

Let C be a nonempty, closed, and convex subset of $\mathcal{H}$. For each point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \qquad \forall y \in C.$$

The mapping $P_C : \mathcal{H} \to C$ is called the metric projection of $\mathcal{H}$ onto C and is characterized by the following two properties: $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x)\rangle \leq 0, \qquad \forall x \in \mathcal{H}, y \in C.$$

We recall the following definitions concerning operator $T : \mathcal{H} \to \mathcal{H}$.

2.1. Definition. The operator $T : \mathcal{H} \to \mathcal{H}$ is called:

• Firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \qquad \forall x, y \in \mathcal{H},$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - Tx) - (y - Ty)\|^2, \qquad \forall x, y \in \mathcal{H}.$$

• Demicontractive if $Fix(T) \neq \emptyset$ and there exists $\beta \in [0, 1)$ such that

$$\|Tx - x^\star\|^2 \leq \|x - x^\star\|^2 + \beta\|x - Tx\|^2, \qquad \forall x \in \mathcal{H}, \quad \forall x^\star \in Fix(T).$$

This is equivalent to

$$\langle x - x^\star, x - Tx\rangle \geq \frac{1-\beta}{2}\|x - Tx\|^2, \qquad \forall x \in \mathcal{H}, \quad \forall x^\star \in Fix(T).$$

• Strict pseudo-contraction, if there exists a constant $\beta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta\|(x - Tx) - (y - Ty)\|^2, \qquad \forall x, y \in \mathcal{H}.$$

Note that every β-strict pseudo-contraction mapping with nonempty fixed point set is β-demicontractive. Also a β-demicontractive mapping is $\frac{2}{1-\beta}$-generalized demimetric. We give an example of a generalized demimetric mapping which is not demicontractive.

2.2. Example. Let $\mathcal{H} = \mathbb{R}$ be the real line. Define T on $\mathbb{R}$ by $T(x) = 3x - 1$. Clearly, $x^\star = \frac{1}{2}$ is the only fixed point of T. We have T is (-2)-generalized demimetric mapping. Indeed, for each $x \in \mathbb{R}$ we have

$$(-2)(x - \frac{1}{2})(1 - 2x) = \zeta\langle x - x^\star, x - Tx\rangle = \|x - Tx\|^2 = (1 - 2x)^2.$$

Putting $x^\star = \frac{1}{2}$ and $x = 1$, we see that T is not demicontractive mapping.

2.3. Definition. Assume that $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping with $Fix(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in $\mathcal{H}$, the following implication holds:

$$x_n \rightharpoonup x \quad \text{and} \quad (I - T)x_n \to 0 \Rightarrow x \in Fix(T).$$

It is known that if $T : \mathcal{H} \to \mathcal{H}$ is a β-strict pseudo-contraction mapping, then $I - T$ is demiclosed at zero.

2.4. Lemma. *[6] Let $\mathcal{H}$ be a real Hilbert space and let θ be a real number with $\theta \neq 0$. Let $T : \mathcal{H} \to \mathcal{H}$ be a θ-generalized demimetric mapping. Then the fixed point set $Fix(T)$ of T is closed and convex.*

2.5. Lemma. *[2] Let C be a nonempty closed and convex subset of real Hilbert space $\mathcal{H}$. Let $F : C \to \mathcal{H}$ be a strongly monotone and Lipschitz continuous mapping. Then $VIP(C, F)$ consists only one point.*

Let B be a mapping of $\mathcal{H}$ into $2^{\mathcal{H}}$. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in \mathcal{H} : Bx \neq \emptyset\}$. A multi-valued mapping B on $\mathcal{H}$ is said to be monotone if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$ and $v \in By$. A monotone mapping B on $\mathcal{H}$ is said to be maximal if its graph is not properly contained in the graph of any other monotone mapping on $\mathcal{H}$. For a maximal monotone mapping B on $\mathcal{H}$ and $r > 0$, we may define a single-valued mapping $J_r = (I + rB)^{-1} : \mathcal{H} \to D(B)$, which is called the resolvent of B for r. Let B be a maximal monotone mapping on $\mathcal{H}$ and let $B^{-1}0 = \{x \in \mathcal{H} : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = Fix(J_r)$ for all $r > 0$; see [10] for more details.

3. Main Result

In this section, we propose our algorithm and subsequently analyzes their convergence properties under some certain conditions.

3.1. Theorem. *Let $\mathcal{H}$ be a Hilbert space. Let for each $i \in \{1, 2, ..., m\}$, $\zeta^{(i)} \neq 0$ and $T^{(i)} : \mathcal{H} \to \mathcal{H}$ be a $\zeta^{(i)}$-generalized demimetric mapping such that $I - T^{(i)}$ is demiclosed at 0. Let for each $i = 1, 2, ..., m$, $A^{(i)} : \mathcal{H} \to \mathcal{H}$ be a maximal monotone operator. Suppose that $\Omega = \bigcap_{i=1}^{m}(Fix(T^{(i)}) \bigcap (A^{(i)})^{-1}(0)) \neq \emptyset$. Let the operator $F : \mathcal{H} \to \mathcal{H}$ be l-Lipschitz continuous and δ-strongly monotone with constants $l > 0$, $\delta > 0$. Assume that $\gamma \in (0, \frac{2\delta}{l^2})$ and $\alpha^* > 0$. Let $\{x_n\}$ be a sequence defined by:*

$$(3) \qquad \begin{cases} x_1, x_0 \in \mathcal{H} \text{ is chosen arbitrarily}, \\ w_n = x_n + \alpha_n^*(x_n - x_{n-1}), \\ y_n = S_n^{(m)} S_n^{(m-1)} ... S_n^{(1)} w_n \\ x_{n+1} = (I - \gamma\beta_n F) y_n, \qquad n \geq 1, \end{cases}$$

where $S_n^{(i)} = J_{r_n^{(i)}}^{A^{(i)}}(I + l^{(i)}\theta_n^{(i)}(T^{(i)} - I))$, $l^{(i)} = \frac{\zeta^{(i)}}{|\zeta^{(i)}|}$, and $0 \leq \alpha_n^* \leq \overline{\alpha}_n$ such that

$$(4) \qquad \overline{\alpha}_n = \begin{cases} min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \alpha^*\}, & x_n \neq x_{n-1} \\ \alpha^*, & otherwise. \end{cases}$$

Let the sequences $\{\beta_n\}, \{\varepsilon_n\}, \{\theta_n^{(i)}\}$ and $\{r_n^{(i)}\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(ii) $\liminf_n r_n^{(i)} > 0$ *for* $i = 1, 2, ..., m$;

(iii) $0 < c^{(i)} \leq \theta_n^{(i)} < 2\frac{l^{(i)}}{\zeta^{(i)}}$, *for* $i \in \{1, 2, ..., m\}$;

(iv) $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$.

Then, the sequence $\{x_n\}$ converges strongly to the unique solution $x^\star \in VI(\Omega, F)$.

References

[1] Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)

[2] Yamada, I.: The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings. In: Butnariu, D., Censor, Y., Reich, S. (eds.) Inherently Parallel Algorithms for Feasibility and Optimization and Their Applications, pp. 473-504. Elsevier, Amsterdam (2001)

[3] I. Iiduka.: Fixed point optimization algorithms for distributed optimization in network systems, SIAM J. Optim., 23 (2013), 1-26.

[4] Zhou, H., Wang, P.: A simpler explicit iterative algorithm for a class of variational inequalities in Hilbert spaces. J. Optim. Theory Appl. 161, 716-727 (2014)

[5] Hieu, D.V., Anh, P.K., Muu, L.D.: Modified hybrid projection methods for finding common solutions to variational inequality problems. Comput. Optim. Appl. 66(1), 75-96 (2017)

[6] Kawasaki T, Takahashi W.: A strong convergence theorem for countable families of nonlinear nonself mappings in Hilbert spaces and applications. J Nonlinear Convex Anal. **19** 543-560 (2018).

[7] Eslamian, M.: Strong convergence theorem for common zero points of inverse strongly monotone mappings and common fixed points of generalized demimetric mappings, Optimization, doi/10.1080/02331934.2021.1939341 (2021).

[8] Alvarez F. Attouch H.: An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Var Anal. 2001;9(1.2):3-11.

[9] Shehu Y, Iyiola OS, Ogbuisi FU.: Iterative method with inertial terms for nonexpansive mappings: applications to compressed sensing. Numer Algorithms. 2020;83(4):1321-1347.

[10] Takahashi, W.: Nonlinear Functional Analysis-Fixed Point Theory and Its Applications. Yokohama Publishers, Yokohama (2000)

E-mail: eslamian@mazust.ac.ir

E-mail: mhmdeslamian@gmail.com