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SOME THEORETICAL RESULTS ON FRACTIONAL OPTIMAL CONTROL PROBLEM

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Abstract. This paper focuses on the existence of at least one solution to the fractional linear quadratic optimal control problem in Caputo sense. The method proposed is based on the Leray-Schauder nonlinear alternative.

1. Introduction

Theory and application of optimal control have been widely used in different fields such as biomedicine, aircraft systems, robotic, etc. However, optimal control of nonlinear systems is a challenging task which has been studied extensively for decades. In the past two decades, the indirect methods have been extensively developed. It is well-known that the nonlinear optimal control problem (OCP) leads to a nonlinear two-point boundary value problem (TPBVP) or a Hamilton-Jacobi-Bellman (HJB) partial differential equation. Many recent researches have been devoted to solve these two problems [4].

In this paper, we consider a nonlinear control system described by:

$$ {}_0^cD_t^\alpha x(t) = Ax(t) + Bu(t), \; x(0) = x_0, \tag{1.1}$$

where $x \in \mathbb{R}^n$ is a state vector; $u \in \mathbb{R}^s$ is a control signal, and ${}_0^cD_t^\alpha$ denotes for the α-th left Caputo fractional derivative with $0 < \alpha \leq 1$. The objective is to find the optimal control

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law $u^*(t)$, which minimizes the following cost function:

$$\frac{1}{2}x(t_f)^T Sx(t_f) + \frac{1}{2}\int_0^{t_f} (x(t)^T Px(t) + 2x(t)^T Qu(t) + u(t)^T Ru(t))\mathrm{d}t, \tag{1.2}$$

where $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times s}$. When $\alpha = 1$, the above problem is converted to the standard linear quadratic optimal control problem. Additionally, $S \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{R}^{n\times n}$ are two symmetric positive semi-definite matrices, $R \in \mathbb{R}^{s\times s}$ is a symmetric positive definite matrix, and $Q \in \mathbb{R}^{n\times s}$ is a real matrix.

2. Optimality conditions

The authors in [5] introduced the necessary and sufficient optimality conditions for the FLQOCP (1) as follows:

$$\begin{cases} {}_0^cD_t^\alpha x(t) = \frac{\partial \mathcal{H}}{\partial \lambda}(t, x, u, \lambda), \\ {}_tD_{t_f}^\alpha \lambda(t) = \frac{\partial \mathcal{H}}{\partial x}(t, x, u, \lambda), \\ \frac{\partial \mathcal{H}}{\partial u}(t, x, u, \lambda) = 0, \\ {}_tI_{t_f}^{1-\alpha}\lambda(t_f) = \frac{\partial \theta}{\partial x}(t_f, x(t_f)), \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $\theta(t, x(t)) = \frac{1}{2}x(t)^T Sx(t)$, and $\mathcal{H}$ denotes the Hamiltonian function defined by

$$\mathcal{H}(t, x, u, \lambda) = \frac{1}{2}\Big(x(t)^T Px(t) + 2x(t)^T Qu(t) + u(t)^T Ru(t)\Big) + \lambda^T(t)(Ax(t) + Bu(t)). \tag{2.2}$$

from the assumption $\frac{\partial \mathcal{H}}{\partial u}(t, x, u, \lambda) = 0$, the exact optimal control is explicitly computed in terms of the vectors $x(t)$, and $\lambda(t)$ as

$$u^*(t) = -R^{-1}Q^T x(t) - R^{-1}B^T\lambda(t). \tag{2.3}$$

By substituting (2.3) into (2.1) and taking the operators ${}_0I_t^\alpha$, and ${}_tI_{t_f}^\alpha$ from both sides of these equations together with the initial condition $x(0) = x_0$, it can be obtained the following coupled dynamic system corresponding to the original minimization problem (1):

$$\begin{cases} x(t) = x_0 + {}_0I_t^\alpha\Big((A - BR^{-1}Q^T)x(t) - (BR^{-1}B^T)\lambda(t)\Big), \\ \lambda(t) = \frac{(t_f-t)^{\alpha-1}}{\Gamma(\alpha)}Sx(t_f) + {}_tI_{t_f}^\alpha\Big((P - QR^{-1}Q^T)x(t) + (-QR^{-1}B^T + A^T)\lambda(t)\Big). \end{cases} \tag{2.4}$$

Note that, if the matrix S is not zero, then the term $\frac{1}{\Gamma(\alpha)}(t_f - t)^{\alpha-1}Sx(t_f)$ is unbounded at $t = t_f$. Thus, in practical use, we take $t \in [0, t_f - \epsilon]$, where ϵ is a chosen sufficiently small constant.

3. Some results on the existence and uniqueness a solution

We saw that the system of Eq. (2.4) has the same solution as the original problem (1). In this section, we obtain the existence of at least one solution for the system (4), using the Leray-Schauder nonlinear alternative.

In this part we want to speak about the existence and uniqueness of a solution to FLQOCP (1.1). We saw that the system of Eq. (2.4) has the same solution as the original problem (1); that is, if $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix}$ is a solution of the equation (2.4), then $\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$ is a solution of the equation (1.1)-(1.2), and vice versa.

We obtain the existence of at least one solution for the system of Eqs. (2.4) using the Leray-Schauder nonlinear alternative. In addition, using the Banach contraction principle, we establish sufficient condition for the uniqueness of the solution.

Theorem 3.1. *(The Leray-Schauder nonlinear alternative)* [6] *Let $\mathbb{X}$ be a Banach space, $\mathcal{C} \subset \mathbb{X}$ be a convex set and $\mathbb{U}$ be open in $\mathcal{C}$ with $0 \in \mathbb{U}$. Let $\mathbb{T} : \bar{\mathbb{U}} \to \mathcal{C}$ be a continuous and compact mapping. Then either*

(1) *The mapping $\mathbb{T}$ has a fixed point in $\mathbb{U}$, or*
(2) *There exists a point $z \in \partial\mathbb{U}$ such that $z = \mu\mathbb{T}z$ for some $\mu \in (0, 1)$.*

Let $\mathcal{C} = C([0, t_f - \epsilon], \mathbb{R}^n) \times C([0, t_f - \epsilon], \mathbb{R}^n)$ in which $C([0, t_f - \epsilon], \mathbb{R}^n)$ is the Banach space of all continuous functions from $[0, t_f - \epsilon]$ into $\mathbb{R}^n$, and ϵ is a given sufficiently small constant. We remark that replacing the interval $[0, t_f]$ by $[0, t_f - \epsilon]$ avoids unboundedness of $(t_f - t)^{\alpha-1}$ at t_f.

Consider the system of equations (2.4). For the sake of brevity, we denote

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T,\ \lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t))^T,$$

$$\phi_1(t) = \Big(x_1(0), x_2(0), \dots, x_n(0)\Big)^T,\ \phi_2(t) = \frac{(t_f - t)^{\alpha-1}}{\Gamma(\alpha)} Sx(t_f),$$

$$J_1(s, x(s), \lambda(s)) = (A - BR^1Q^T)x(s) - (BR^{-1}B^T)\lambda(s),$$

$$J_2(s, x(s), \lambda(s)) = (P - QR^{-1}Q^T)x(s) + (-QR^{-1}B^T + A^T)\lambda(s).$$

Let $K_1(t, s)$ and $K_2(t, s)$ be two $n \times n$ diagonal matrices with all off-diagonal entries equal to 0, and diagonal entries as

$$[K_1(t,s)]_{ii} = \begin{cases} \frac{1}{(t-s)^{1-\alpha}}, & (t,s) \in \Delta, \\ 0, & \text{elsewhere}, \end{cases} \quad \text{i} = 1, 2, \dots, \text{n},$$

$$[K_2(t,s)]_{ii} = \begin{cases} \frac{1}{(s-t)^{1-\alpha}}, & (t,s) \in \Delta', \\ 0, & \text{elsewhere}, \end{cases} \quad \text{i} = 1, 2, \dots, \text{n}.$$

in which $\Delta := \{(t, s) :\ 0 \le s < t \le t_f\}$, and $\Delta' := \{(t, s) :\ 0 \le t < s \le t_f\}$. Defining

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix},\ \Phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix},\ K(t,s) = \begin{pmatrix} K_1(t,s) & 0 \\ 0 & K_2(t,s) \end{pmatrix},$$

$$J(s, \mathbf{X}(s)) = \begin{pmatrix} J_1(s, x(s), \lambda(s)) \\ J_2(s, x(s), \lambda(s)) \end{pmatrix},\ H = \begin{pmatrix} A - BR^1Q^T & -BR^{-1}B^T \\ P - QR^{-1}Q^T & -QR^{-1}B^T + A^T \end{pmatrix},$$

The equation (2.4) is transformed to the following integral system:

$$\mathbf{X}(t) = \Phi(t) + \int_0^{t_f} K(t,s)J(s,\mathbf{X}(s))\mathrm{d}s. \tag{3.1}$$

In view of (3.1), the system (2.4) is converted to a fixed point problem as follows:

$$\mathbf{X}(t) = \mathbf{T}\mathbf{X}(t),$$

where $\mathbf{T} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\mathbf{T}\mathbf{X}(t) = \Phi(t) + \int_0^{t_f} K(t,s)J(s,\mathbf{X}(s))\mathrm{d}s. \tag{3.2}$$

We observe that the system of equations (2.4) has a solution if and only if the operator $\mathbf{T}$ has a fixed point. The following theorem allows us to deduce the existence of a solution to the problem (1.1).

Theorem 3.2. *The original problem (1.1)-(1.2) has a unique solution on* $[0, t_f-\epsilon]$*, provided* $\frac{\|H\|}{\alpha}t_f^\alpha < 1$*, in which* $\|H\|$ *is any matrix operator norm for the matrix* H*, and* ϵ *is a given sufficiently small constant.*

Proof. Let $B_r = \{\mathbf{X} \in \mathcal{C} : \ \|\mathbf{X}\| < r\}$, with $r = \frac{\|\Phi\|}{1-\frac{\|H\|}{\alpha}t_f^\alpha}$, $\|\mathbf{X}\| = \sup_{0\le t\le t_f-\epsilon}\{|x(t)|, |\lambda(t)|\}$, and $\|\Phi\| = \sup_{0\le t\le t_f-\epsilon}\{|\phi_1(t)|, |\phi_2(t)|\}$.

Step 1: Transforming $\overline{B}_r$ into itself by the operator $\mathbf{T}$

For each $\mathbf{X} \in \overline{B}_r$, we obtain

$$\begin{aligned}
|\mathbf{T}\mathbf{X}(t)| &\le |\Phi(t)| + \int_0^{t_f-\epsilon} |K(t,s)||J(s,\mathbf{X}(s))|\mathrm{d}s \\
&\le |\Phi(t)| + \int_0^{t_f-\epsilon} |K(t,s)||H\mathbf{X}(s)|\mathrm{d}s \\
&\le |\Phi(t)| + \int_0^{t_f-\epsilon} |K(t,s)|\|H\||\mathbf{X}(s)|\mathrm{d}s \\
&\le |\Phi(t)| + \|H\|\|\mathbf{X}\| \int_0^{t_f-\epsilon} |K(t,s)| \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \mathrm{d}s \\
&\le \|\Phi\| \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} + \frac{t_f^\alpha}{\alpha}\|H\|\|\mathbf{X}\| \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix},
\end{aligned}$$

where $\mathbf{1} = (\underbrace{1, 1, \ldots, 1}_{\text{n times}})^T$. Therefore,

$$\|\mathbf{T}\mathbf{X}\| \le \|\Phi\| + \frac{t_f^\alpha}{\alpha}\|H\|\|\mathbf{X}\| \le r, \tag{3.3}$$

which means that $\mathbf{T} : \overline{B}_r \to \overline{B}_r$.

Step 2: The contractivity of the operator $\mathbf{T}$

Let $\mathbf{X}_1, \mathbf{X}_2 \in \overline{B}_r$. Then

$$\begin{aligned}
|\mathbf{T}\mathbf{X}_1(t) - \mathbf{T}\mathbf{X}_2(t)| &\leq \Big| \int_0^{t_f-\epsilon} K(t,s)\Big(J(s,\mathbf{X}_1(s)) - J(s,\mathbf{X}_2(s))\Big)\mathrm{d}s\Big| \\
&\leq \int_0^{t_f-\epsilon} |K(t,s)||H(\mathbf{X}_1(s) - \mathbf{X}_2(s))|\mathrm{d}s \\
&\leq \|H\|\|\mathbf{X}_1 - \mathbf{X}_2\| \int_0^{t_f-\epsilon} |K(t,s)| \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \mathrm{d}s \\
&\leq \frac{t_f^\alpha}{\alpha}\|H\|\|\mathbf{X}_1 - \mathbf{X}_2\| \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}.
\end{aligned}$$

So,

$$\|\mathbf{T}\mathbf{X}_1 - \mathbf{T}\mathbf{X}_2\| \leq \frac{t_f^\alpha}{\alpha}\|H\|\|\mathbf{X}_1 - \mathbf{X}_2\|.$$

That is $\mathbf{T}$ is a contractive operator since $\frac{\|H\|}{\alpha}t_f^\alpha < 1$. By the Banach contraction principle, $\mathbf{T}$ has a unique fixed point on $[0, t_f - \epsilon]$, which is the solution of the original problem (1.1)-(1.2). □

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