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CONVERGENT ANALYTICAL HOMOTOPY PERTURBATION METHOD SOLUTION OF SET OF MULTIVARIABLE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, Homotopy Perturbation Method (HPM) is applied to solve two nonlinear partial differential equations (PDE) of Burger-Fisher equation and Witham-Broer-Kaup equations. These are such cases of set of multivariable and coupled nonlinear dynamics. The validity of HPM solution is assessed by comparing the exact solutions together with the Variational Iteration Method (VIM) and Homotopy Analysis Method (HAM). Result signifies the efficiency of the HPM to solve nonlinear PDEs.

1. Introduction

The aim of this paper is to illustrate the efficiency of HPM [1] to solve nonlinear PDEs. Accordingly, the method is applied to two well-known equations. Asymptotic solutions are then compared with exact solutions as well as the Variational Iteration Method (VIM) [2] and Homotopy Analysis Method (HAM) [3]. Finally, the work will be concluded at section 3.

2. Illustrative Examples

Consider the generalized Burger-Fisher equation [4] as a model for propagation of a mutant gene with $u(x,t)$ displaying the density with α, β, λ, and σ are parameters [5]:

Key words and phrases. Homotopy perturbation method, Witham-Broer-Kaup equations, Burger-Fisher equation.

* Speaker.

$$u_t + \alpha u^\sigma u_x - \lambda u_{xx} = \beta u(1 - u^\sigma), \tag{2.1}$$

where σ is a positive integer. Choosing $\alpha = 1$, $\sigma = 2$, $\lambda = 1$, and $\beta = 1$ converts eq. (1) to:

$$u_t + u^2 u_x - u_{xx} = u(1 - u^2), t \geq 0, 0 \leq x \leq 1. \tag{2.2}$$

The exact solution, for the given initial condition $u(x, 0)$, is given as follows:

$$u(x, 0) = (\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{3}x))^{\frac{1}{2}}, \tag{2.3}$$

$$u(x, t) = (\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{3}x - \frac{10}{9}t))^{\frac{1}{2}}. \tag{2.4}$$

The analytical solution and that of by HAM can be observed in [4]. To apply HPM, using the basic concept of the method in [1], eq. (2) is separated into two linear and nonlinear parts, which are as follow:

$$L(u) = u_t, N(u) = u^2 u_x - u_{xx} - u + u^3. \tag{2.5}$$

The homotopy statement is written as follows, assuming p as an embedding parameter:

$$h(v, p) = v_t - L(v_t) + pL(v_0) + p(u^2 u_x - u_{xx} - u + u^3) = 0. \tag{2.6}$$

Choosing initial guess as $L(v_0) = 0$ and substituting eq. (5) into eq. (6) and rearranging the result in terms of ascending powers of p, results the following equations as:

$$p^0(v_{0t}) = 0, \tag{2.7}$$

$$p^1(v_{1t} - v_{0xx} - v_0 + v_0^2 v_{0x} + v_0^3) = 0, ... \tag{2.8}$$

Applying the initial condition (eq. (3)), which satisfies eq. (7), and then solving the above equations, the following results will be obtained:

$$v_0 = (\frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{3}))^{\frac{1}{2}}, \tag{2.9}$$

$$v_1 = \frac{0.5556}{\sqrt{2 - 2\tanh(\frac{x}{3})}}(\tanh^2(\frac{x}{3} - 1)t, .. \tag{2.10}$$

$$V = v_1 + v_2 + v_3 + v_4 + v_5. \tag{2.11}$$

To the purpose of summarization, the last three terms of eq. (11) are not stated. The efficiency of the proposed HPM solution can be seen in fig. (1).

Similarly, Whitham [6]-Broer [7]-Kaup [8] (WBK) equations will be considered:

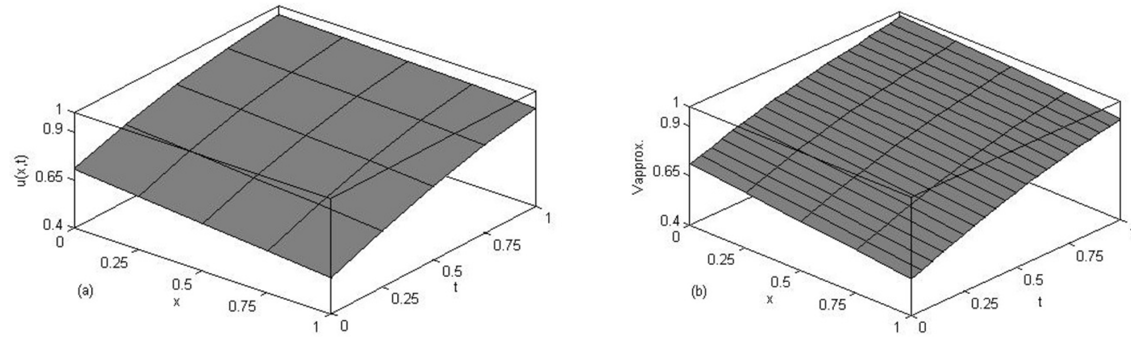

FIGURE 1. (a) the exact solution and (b) the HPM solution

$$
\begin{aligned}
v_t + (uv)_x + \alpha u_{xxx} - \beta v_x &= 0,\\
u_t + uu_x + v_x + \beta u_{xx} &= 0,
\end{aligned} \tag{2.12}
$$

where $u = u(x,t)$ is the horizontal velocity, $v = v(x,t)$ is the height that deviates from equilibrium position of the liquid, and α and β are constants which are represented in different diffusion powers. Dynamic (12) is a model to describe dispersive waves. If $\alpha = 0$ and $\beta \neq 0$, then the equation represents the classical long wave equation that describes shallow water wave with dispersion. If $\alpha = 1$ and $\beta = 0$, then the dynamic represents the variant Boussinesq equation. In [2], VIM is applied to solve WBK equations with the initial conditions:

$$
\begin{aligned}
u(x,0) &= \lambda - 2Bk\coth(k\zeta),\\
v(x,0) &= -2B(B+\beta)k^2 csch^2(k\zeta),
\end{aligned} \tag{2.13}
$$

where $B = \sqrt{\alpha + \beta^2}$, $\zeta = x + x_0$, and x_0, k, and λ are arbitrary constants. To find the HPM solution, linear $L(u)$ and nonlinear $N(u)$ parts, together with the homotopy statement $h(.,p)$, are distinguished as:

$$
\begin{aligned}
L(u) = u_t, N(u) &= uu_x + v_x + \beta u_{xx},\\
L(v) = v_t, N(v) &= (uv)_x + \alpha u_{xxx} - \beta v_{xx}.
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
h(u,p) &= u_t - L(u_0) + pL(u_0) + p(uu_x + v_x + \beta u_{xx}) = 0,\\
h(v,p) &= v_t - L(v_0) + pL(v_0) + p((uv)_x + v_x + \alpha u_{xxx} - \beta v_{xx}) = 0.
\end{aligned} \tag{2.15}
$$

Setting initial guess at zero, substituting eq. (14) in eq. (15) and then rearranging with respect to ascending powers of p results the following equations:

$$
\begin{aligned}
p^0(u_{0t}) &= 0,\\
p^0(v_{0t}) &= 0.
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
p^1(u_{1t} + u_0 u_{0x} + v_{0x} + \beta u_{0xx}) &= 0,\\
p^1(v_{1t} + u_{0x} v_0 + u_0 v_{0x} + \alpha u_{0xxx} - \beta v_{0xx}) &= 0.
\end{aligned} \tag{2.17}
$$

The results are as follows while $k = 0.1$, $\lambda = 0.005$, $\alpha = 1.5$, $\beta = 1.5$, and $x_0 = 10$.

$$
\begin{aligned}
u_0 &= 0.0050 - 0.3872 \coth(\frac{x}{10}\\
v_0 &= -0.1330 csch^2(\frac{x}{10}
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
u_1 &= -frac0.6000 \times 10^{-7} t(3223 \sinh(\frac{x}{10} + 1) + 256 \cosh(\frac{x}{10} + 1))(\cosh^2(\frac{x}{10} + 1) - 1) \sinh(\frac{x}{10}\\
v_1 &= -frac{+}2645 + 0.2000 \times 10^{-7} t(24 \cosh^2(\frac{x}{10} + 1) + 6650 \cosh(\frac{x}{10} + 1) \sinh(\frac{x}{10}) + 1)(\cosh^2(\frac{x}{10} + 1)
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
U &= u_1 + u_2\\
V &= v_1 + v_2
\end{aligned} \tag{2.20}
$$

The accuracy of the solutions is demonstrated in tab. (1), which contains the absolute errors by HPM and VIM. The absolute errors for VIM are derived from [2].

3. Conclusion

In this paper, the efficiency of the homotopy perturbation method to solve nonlinear partial differential equations is verified. The solution is compared with the exact solutions and those produced by VIM and HAM. The results confirmed the significance of HPM with respect to the exact solution. The use of HPM in nonlinear dynamics is more promising, when a typical application i.e. Burger-Fisher and Whitham-Broer-Kuap equations, is shown here. The results confirm the capability of HPM as a powerful tool to solve nonlinear PDEs.

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TABLE 1. Comparing exact results with those of HPM and VIM

(x,t)	$\lvert u_{exact} - U_{HPM}\rvert$	$\lvert u_{exact} - u_{VIM}\rvert$	$\lvert u_{exact} - V_{HPM}\rvert$	$\lvert u_{exact} - v_{VIM}\rvert$
(0.1,0.1)	119 E-04	13033E-04	1.073 E-04	1.10430E-04
(0.1,0.3)	3..606 E-04	3.69597E-04	3.042 E-04	3.31865E-04
(0.1,0.5)	5.940 E-04	6.16873E-04	4.731 E-04	5.54071E-04
(0,0.1)	1.190 E-04	1.19869E-04	1.041 E-04	1.07016E-04
(0,0.3)	3.528 E-04	3.60098E-04	2.958 E-04	31601E-04
(0,0.5)	5.826 E-04	6.01006E-04	4.615 E-04	5.36927E-04
(0.3,0.1)	1.161 E-04	1.16789E-04	1.010 E-04	1.03737E-04
(0.3,0.3)	3.451 E-04	3.50866E-04	2.876 E-04	3.11737E-04
(0.3,0.5)	5.713 E-04	5.85610E-04	4.502 E-04	50447E-04
(0.4,0.1)	1.133 E-04	1.13829E-04	0.980 E-04	1.00579E-04
(0.4,0.3)	3.376 E-04	3.41948E-04	2.797 E-04	3.02245E-04
(0.4,0.5)	5.601 E-04	5.70710E-04	4.391 E-04	5.04593E-04
(0.5,0.1)	1.105 E-04	1.10936E-04	0.951 E-04	9.75385E-05
(0.5,0.3)	3.302 E-04	3.33274E-04	2.721 E-04	2.93107E-04
(0.5,0.5)	5.490 E-04	5.56235E-04	482 E-04	4.89335E-04

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