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σ-CONVEX FUNCTIONS AND THEIR PROPERTIES

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ABSTRACT. We introduce and study the notion of σ-convex functions. We show that many well known properties of the convex function, namely Lipschitz property in the interior of its domain, remain valid for the large class of σ-convex functions.

1. Introduction

Suppose that X is a Banach Space with topological dual space X^*. We will denote by $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ the duality pairing between X and X^*. Also we will denote by $\mathbb{R}_+$ the real nonnegative numbers.

Let T be a set-valued map from X to X^*. We recall that T is monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $x, y \in X$ and $x^* \in T(x), y^* \in T(y)$.

The domain and graph of T are, respectively, defined by

$$D(T) = \{x \in X : T(x) \neq \emptyset\},$$

$$\operatorname{gr} T = \{(x, x^*) \in X \times X^* : x \in D(T), \text{ and } x^* \in T(x)\}.$$

For two multivalued operators T and S we write $T \subseteq S$ if S is an extension of T, i.e., $\operatorname{gr} T \subseteq \operatorname{gr} S$. A monotone operator is called maximal monotone if it has no monotone extension other than itself. For the history of monotone operators see [4].

First we recall the following definition from [3, 5].

Definition 1.1. (i) Given an operator $T : X \to 2^{X^*}$ and a map $\sigma : D(T) \to \mathbb{R}_+$, T is said to be σ-*monotone* if for every $x, y \in D(T)$, $x^* \in T(x)$ and $y^* \in T(y)$,

$$\langle x^* - y^*, \ x - y \rangle \geq -\min\{\sigma(x), \sigma(y)\} \|x - y\|. \tag{1.1}$$

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(ii) A σ-monotone operator T is called *maximal σ-monotone*, if for every operator T' which is σ'-monotone with $\operatorname{gr} T \subseteq \operatorname{gr} T'$ and σ' an extension of σ, one has $T = T'$.

For more information about σ-monotonicity and maximal σ-monotonicity, we refer to the papers [1], [2],[3] and [5].

Note that when $\sigma(x) = \epsilon$, (ϵ is a constant positive real number) the definition of σ-monotonicity reduces to ϵ-monotonicity [6].

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Its domain (or effective domain) is defined by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$. The function f is called proper if $\operatorname{dom} f \neq \emptyset$. In addition, f is said to be convex if for all $x, y \in X$ and for each $t \in]0, 1[$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \qquad \forall y \in X\}$$

and $\partial f(x)$ is empty if x is not in the domain of f.

The notion of σ-convex function introduced in [2] and studied its relation with σ-monotonicity. The notion of σ-convexity covers the concepts of ϵ-convexity [6, 7] and convexity. The convex functions are central to the study of Convex Analysis and Optimization.

2. Main results

We recall from [6] that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is ε-convex if it satisfies the following inequality for every $a, b \in X$, and $\lambda \in]0, 1[$

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b) + \lambda(1-\lambda)\varepsilon||a - b||.$$

In [7], Luc, Ngai and Thera presented several properties of ε-convex functions, and studied relationships between ϵ-convexity and ϵ-monotonicity. The connection between ϵ-subdifferential and ϵ-monotonicity was investigated in [6] by Jofre, Luc and Thera. Also for a historical note about the ϵ-convexity and ϵ-monotonicity, we refer the reader to [6].

The notion of σ-convexity is introduced and studied in [2]. We recall it here:

Definition 2.1. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a map σ form $\operatorname{dom} f$ to $\mathbb{R}_+$, we say that f is *σ-convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t(1-t)\min\{\sigma(x), \sigma(y)\}||x - y|| \tag{2.1}$$

for all $x, y \in X$, and $t \in]0, 1[$.

A special case of σ-convex functions are the ε-convex functions: these are functions for which $\sigma(x) = \varepsilon \geq 0$ for all $x \in \operatorname{dom} f$. There are σ-convex functions which are not ε-convex for any $\varepsilon \geq 0$, as shown in the following example, taken from [2].

Example 2.2. Consider the functions $\varphi, f, \sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x \sin^2 x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

$$\sigma(x) = \max \left\{ \varphi(x), \max_{z \leq x} \varphi(z) - \varphi(x) \right\}$$

$$f(x) = \int_0^x \varphi(t) dt.$$

It follows from Example 3.7 in [2] that f is σ-convex, but it is not ϵ-convex for any $\epsilon > 0$.

Note that if f is a σ-convex function, then $\operatorname{dom} f$ is a convex set.
Some elementary properties of σ-convex functions are given below:

Proposition 2.3. (i) *Suppose that f_1 and f_2 are σ_1-convex and σ_2-convex, respectively, with* $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$ *and $\alpha > 0$. Then $\alpha f_1 + f_2$ is $(\alpha\sigma_1 + \sigma_2)$-convex.*
(ii) *If f is σ-convex and $\sigma \leq \sigma'$, then f is σ'-convex.*
(iii) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Then f is σ-convex if and only if for all $x, y \in X$, and $t \in]0, 1[$,*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t(1-t)\sigma(x)||x - y|| \tag{2.2}$$

or equivalently,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t(1-t)\sigma(y)||x - y||$$

(iv) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Then f is convex if and only if it is σ-convex for every $\sigma : \operatorname{dom} f \to \mathbb{R}_+$.

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we define the map $\sigma_f : \operatorname{dom} f \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\sigma_f(x) = \inf\{a \in \mathbb{R}_+ : \frac{f(tx + (1-t)y) - tf(x) - (1-t)f(y)}{t(1-t)} \leq a\|x - y\|, \forall y \in \operatorname{dom} f, t \in]0, 1[\}.$$

It should be noticed that if f is σ'-convex for some $\sigma' : \operatorname{dom} f \to \mathbb{R}_+$, then

$$\sigma_f = \inf\{\sigma : f \text{ is } \sigma\text{-convex}\}. \tag{2.3}$$

In this case, σ_f is finite and f is σ_f-convex. Note that σ_f is the minimal σ such that f is σ-convex.

In the next proposition we give an explicit formula for σ_f.

Proposition 2.4. *Suppose that f is σ-convex for some σ. Then*

$$\sigma_f(x) = \max \left\{ 0, \sup_{t \in]0,1[} \sup_{y \in \operatorname{dom} f \setminus \{x\}} \frac{f(tx + (1-t)y) - tf(x) - (1-t)f(y)}{t(1-t)\|x - y\|} \right\}. \tag{2.4}$$

Proposition 2.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Then σ_f is finite and f is σ_f-convex if and only if f is σ-convex for some σ.*

Remark 2.6. Let $\{f_i\}_{i\in I}$ be an arbitrary family of σ-convex functions. If $f(x) = \sup_{i\in I} f_i(x)$, $x \in X$,then f is σ-convex. Indeed, for $x, y \in X$ and $t \in]0,1[$ we have

$$\begin{aligned} f_i\left(tx + (1-t)\right)y &\leq tf_i\left(x\right) + (1-t)\, f_i\left(y\right) + t\left(1-t\right)\min\left\{\sigma\left(x\right), \sigma\left(y\right)\right\}\|x-y\| \\ &\leq tf\left(x\right) + (1-t)\, f\left(y\right) + t\left(1-t\right)\min\left\{\sigma\left(x\right), \sigma\left(y\right)\right\}\|x-y\|. \end{aligned}$$

Taking the supremum over $i \in I$ implies σ-convexity of f.

Lemma 2.7. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, σ-convex function. Assume that either X is finite-dimensional, or that f is lsc and X is a Banach space. Then f is locally bounded from above in the interior of its domain.*

Lemma 2.8. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a σ-convex function. Assume that f is bounded from above in a neighborhood of some point x_0. Then f is locally bounded from above in the interior of its domain.*

We introduce the following assumption:

Property B: We say that the function σ has the property B, if for every $x \in \operatorname{int}\operatorname{dom} f$ and every $\varepsilon > 0$ sufficiently small, σ is bounded on the sphere
$S(x,\varepsilon) = \{y \in X : \|x - y\| = \varepsilon\}$.

Note that this assumption is weaker than assuming that σ is locally bounded. For example, the function σ such that $\sigma(x) = 1/\|x\|$ for $x \neq 0$ and $\sigma(0) = 1$, satisfies property B without being locally bounded.

Theorem 2.9. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a σ-convex function. Assume that f is locally bounded from above in the interior of its domain. If σ satisfies property B, then f is locally Lipschitz in the interior of its domain.*

Corollary 2.10. *Every proper, σ-convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its domain.*

Proposition 2.11. *Suppose that $f : I \to \mathbb{R}$ and $\sigma : I \to \mathbb{R}_+$ is a map. Then the following statements are equivalent:*

(i) f is σ-convex;

(ii) For $s, t, u \in I$ with $a < s < t < u < b$,

$$\frac{f\left(t\right) - f\left(s\right)}{t - s} \leq \frac{f\left(u\right) - f\left(t\right)}{u - t} + \min\left\{\sigma\left(s\right), \sigma\left(u\right)\right\}. \tag{2.5}$$

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