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EXISTENCE OF SOLUTIONS FOR THE FRACTIONAL Q-DIFFERENTIAL INCLUSION ON MULTIFUNCTIONS BY APPROXIMATE ENDPOINT PROPERTY

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Abstract. This article by approximate endpoint property, describes a technique for existing of solutions of the fractional q-differential inclusion with boundary value conditions on multifunctions. For to do this, we use an approximate endpoint result on multifunctions. Also, we give an example to elaborate our results and to present the obtained results by fractional calculus.

1. Introduction

Fractional calculus and q-calculus are one of the significant branches in mathematical analysis. In 1910, the subject of q-difference equations introduce by Jackson [6]. In this paper, we are working to stretch out the problem in a sense for the fractional q-differential inclusion problem:

$$ {}^{c}D_q^{\alpha} u(t) \in T\left(t, u(t), u'(t), u''(t)\right), \tag{1.1}$$

with integral boundary conditions:

$$\begin{aligned} &u(0) + u(p) + u(1) = \int_0^1 f_0(s, u(s))\, ds, \\ &{}^{c}D_q^{\beta} u(0) + {}^{c}D_q^{\beta} u(p) + {}^{c}D_q^{\beta} u(1) = \int_0^1 f_1(s, u(s))\, ds, \\ &{}^{c}D_q^{\gamma} u(0) + {}^{c}D_q^{\gamma} u(p) + {}^{c}D_q^{\gamma} u(1) = \int_0^1 f_2(s, u(s))\, ds, \end{aligned} \tag{1.2}$$

where $\alpha \in (2, 3]$, $0 < q, p, \beta < 1$, $\gamma \in (1, 2)$, $f_i : J \times \mathbb{R} \to \mathbb{R}$, here $i = 1, 2, 3$, are continuous functions, $T \colon J \times \mathbb{R}^3 \to P_{cp}(\mathbb{R})$ is a multifunction and ${}^{c}D_q^{\beta}$ is the fractional Caputo type q-derivative for $t \in J = [0, 1]$. The set of all compact subsets of $\mathbb{R}$ denote by $P_{cp}(\mathbb{R})$.

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2. Preliminaries

Assume that $q \in (0,1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [6]. The power function $(a-b)_q^n$ with $n \in \mathbb{N}_0$ is $(a-b)_q^{(n)} = \prod_{k=0}^{n-1}(a-bq^k)$ and $(a-b)_q^{(0)} = 1$ where $a, b \in \mathbb{R}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have $(a-b)_q^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty}(a-bq^k)/(a-bq^{\alpha+k})$. If $b = 0$, then it is clear that $a^{(\alpha)} = a^\alpha$. The q-Gamma function is given by $\Gamma_q(x) = (1-q)^{(x-1)}/(1-q)^{x-1}$, where $x \in \mathbb{R}\backslash\{0, -1, -2, \cdots\}$ [6]. Note that, $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$. The value of q-Gamma function, $\Gamma_q(x)$, for input values q and x with counting the number of sentences n in summation by simplifying analysis. The q-derivative of function f, is defined by $(D_q f)(x) = (f(x) - f(qx))/((1-q)x)$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ [1]. Also, the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [1]. The q-integral of a function f defined on $[0, b]$ is define by $I_q f(x) = \int_0^x f(s) d_q s = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k)$, for $0 \leq x \leq b$, provided that the sum converges absolutely [1]. If $a \in [0, b]$, then $\int_a^b f(u) d_q u = I_q f(b) - I_q f(a) = (1-q)\sum_{k=0}^{\infty} q^k \left[b f(bq^k) - a f(aq^k)\right]$, whenever the series exists. The operator I_q^n is given by $(I_q^0 f)(x) = f(x)$ and $(I_q^n f)(x) = (I_q(I_q^{n-1} f))(x)$ for all $n \geq 1$ [1]. It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [1]. The fractional Riemann-Liouville type q-integral of the function f on $[0, 1]$, of $\alpha \geq 0$ is given by $(I_q^0 f)(t) = f(t)$ and $(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s$, for $t \in [0, 1]$ and $\alpha > 0$ [4]. Also, the fractional Caputo type q-derivative of the function f is given by $\left({}^c D_q^\alpha f\right)(t) = \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^t (t-qs)^{([\alpha]-\alpha-1)} (D_q^{[\alpha]} f)(s) d_q s$, for $t \in J$ and $\alpha > 0$ [4]. It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x)$, and $\left(D_q^\alpha(I_q^\alpha f)\right)(x) = f(x)$, where $\alpha, \beta \geq 0$ [4]. We say a multifunction $G : J \to P_{cl}(\mathbb{R})$ is measurable whenever for each real number y, the function $t \mapsto d(y, G(t))$ is measurable [3]. The Pompeiu-Hausdorff metric $H_d\colon 2^X \times 2^X \to [0, \infty)$ on a metric space (X, ρ) is defined by, $H_\rho(A, B) = \max\{\sup_{a\in A} \rho(a, B), \sup_{b\in B} \rho(A, b)\}$, where $\rho(A, b) = \inf_{a\in A} \rho(a, b)$ [5]. Denote the set of closed and bounded and the set of closed subsets of X by $CB(X)$ and $C(X)$, respectively. In this case $(CB(X), H_\rho)$, $(C(X), H_\rho)$ are a metric space, a generalized metric space, respectively. An element $z \in X$ is called an endpoint of multifunction $T : X \to 2^X$ whenever $Tz = \{z\}$ [2]. Also, multifunction T has approximate endpoint property whenever $\inf_{x\in X} \sup_{y\in Tx} \rho(x, y) = 0$ [2]. A function $\theta : \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} \theta(\lambda_n) \leq \theta(\lambda)$ for all sequence $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \to \lambda$ [2].

3. Main results

Lemma 3.1. *Suppose that $v \in C(J, \mathbb{R})$, $\alpha \in (2, 3]$, $0 < \beta, q, p < 1$, $\gamma \in (1, 2)$ and $f_i : J \times \mathbb{R} \to \mathbb{R}$, here $i = 0, 1, 2$, be continuous functions. The unique solution of the fractional q-differential problem*

$$^cD_q^\alpha u(t) = v(t), \tag{3.1}$$

with conditions (1.2) is given by

$$u(t) = I_q^\alpha v(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))\,ds - \frac{1}{3}\left[I_q^\alpha v(1) + I_q^\alpha v(p)\right] + a_1(t)\int_0^1 f_1(s,u(s))\,ds$$
$$+ a_2(t)\left[I_q^{\alpha-\beta}v(1) + I_q^{\alpha-\beta}v(p)\right] + (b_1 + a_3(t))\int_0^1 g_2(s,u(s))\,ds + (b_2 + a_4(t))\left[I_q^{\alpha-\gamma}v(1) + I_q^{\alpha-\gamma}v(p)\right], \quad (3.2)$$

where $a_1(t) = \frac{3t\Gamma_q(2-\beta)-(p+1)\Gamma_q(2-\beta)}{3(p^{1-\beta}+1)}$, $a_2(t) = \frac{(p+1)\Gamma_q(2-\beta)-3\Gamma_q(2-\beta)t}{3(p^{1-\beta}+1)}$,

$$\begin{aligned}
a_3(t) &= \frac{-6(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)t}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} + \frac{3(p^{1-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(3-\beta)t^2}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)},\\
a_4(t) &= \frac{6(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)t}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} - \frac{3\Gamma_q(3-\gamma)\Gamma_q(3-\beta)(p^{1-\beta}+1)t^2}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)},\\
b_1 &= \frac{2(p+1)(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} - \frac{(p^2+1)\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)},\\
b_2 &= \frac{(p^2+1)\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} - \frac{2(p+1)(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}.
\end{aligned} \quad (3.3)$$

Assume that $\mathcal{X} = C^2(J)$ endowed with the norm $\|u\| = \sup_{t\in J}|u(t)| + \sup_{t\in J}|u'(t)| + \sup_{t\in J}|u''(t)|$. Then $(\mathcal{X}, \|.\|)$ is a Banach space. For $u \in \mathcal{X}$, we define the selection set $S_{T,u}$ by the set of all $v \in L^1(J)$ somehow that $v(t) \in T(t, u(t), u'(t), u''(t))$ for all $t \in\in J$. For the study of problem (1.1) and (1.2), we shall consider the following conditions.

(C1) The multifunction $T : J \times \mathbb{R}^3 \to P_{cp}(\mathbb{R})$ be an integrable bounded such that $T(., x_1, x_2, x_3) : J \to P_{cp}(\mathbb{R})$ is measurable for all $x_i \in \mathbb{R}$;

(C2) The functions $f_i : J \times \mathbb{R} \to \mathbb{R}$ be continuous and map $\theta : [0,\infty) \to [0,\infty)$ be a nondecreasing upper semi-continuous such that $\liminf_{t\to\infty}(t-\theta(t)) > 0$ and $\theta(t) < t$ for all $t > 0$;

(C3) There exist $m, m_0, m_1, m_2 \in C(J, [0,\infty))$ such that $H_d(T(t,x_1,x_2,x_3), T(t,x_1',x_2',x_3')) \leq (m(t)\theta(\sum_{k=1}^3 |x_i - x_i'|))/(\Lambda_1+\Lambda_2+\Lambda_3)$, and $|f_j(t,x) - f_j(t,x')| \leq (m_j(t)\psi(|x-x'|))/(\Lambda_1+\Lambda_2+\Lambda_3)$, for all $t \in J$, $x, x', x_i, x_i' \in \mathbb{R}$, where

$$\Lambda_1 = \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha+1)} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma_q(\alpha+1)} + \frac{5\Gamma_q(2-\beta)\|m_1\|_\infty}{3} + \frac{10\Gamma_q(2-\beta)\|m\|_\infty}{3\Gamma_q(\alpha-\beta+1)}\right.$$
$$\left. + 10\big(2\Gamma_q(2-\beta) + \Gamma_q(3-\beta)\big)\left(\frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty\Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty\right)}{3\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma+1)}\right)\right],$$

$$\Lambda_2 = \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha)} + \frac{2\Gamma_q(2-\beta)\|m\|_\infty}{\Gamma_q(\alpha-\beta+1)} + \big(2\Gamma_q(2-\beta) + \Gamma_q(3-\beta)\big)\left(\frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty\Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty\right)}{\Gamma(3-\beta)\Gamma_q(\alpha-\gamma+1)}\right)\right],$$

$$\Lambda_3 = \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty\Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty\right)}{\Gamma_q(\alpha-\gamma+1)}\right];$$

(C4) Multifunction $N : \mathcal{X} \to 2^{\mathcal{X}}$ is given by $N(u) = \{h \in \mathcal{X} \mid \exists\, v \in S_{T,u} \;:\; h(t) = w(t)\}$, for each $t \in J$, where by applying the notation in (3.3), we have

$$w(t) = I_q^\alpha v(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))ds - \frac{1}{3}\left[I_q^\alpha v(1) + I_q^\alpha v(p)\right] + a_1(t)\int_0^1 f_1(s,u(s))ds$$
$$+ a_2(t)\left[I_q^{\alpha-\beta}v(1) + I_q^{\alpha-\beta}v(p)\right] + (b_1 + a_3(t))\int_0^1 f_2(s,u(s))ds + (b_2 + a_4(t))\left[I_q^{\alpha-\gamma}v(1) + I_q^{\alpha-\gamma}v(p)\right].$$

Theorem 3.2. *The boundary value q-differential inclusion problem (1.1) and (1.2) has a solution, whenever the multifunction $N : \mathcal{X} \to P(\mathcal{X})$ has the approximate endpoint property and consitions (C1)–(C4) are hold.*

Example 3.3. Consider the fractional q-differential inclusion problem

$$ {}^{c}D_q^{\frac{9}{4}}u(t) \in \left[0, \frac{t^2}{100}\sin u(t) + \frac{1}{100}\cos u'(t) + \frac{1}{100}\left(\frac{|u''(t)|}{1+|u''(t)|}\right)\right], \tag{3.4} $$

with the integral boundary conditions

$$ \begin{array}{l} u(0) + u(\frac{3}{4}) + u(1) = \int_0^1 \frac{s^2}{20}\cos u(s)ds, \\ {}^{c}D_q^{\frac{2}{3}}u(0) + {}^{c}D_q^{\frac{2}{3}}u(\frac{3}{4}) + {}^{c}D_q^{\frac{2}{3}}u(1) = \int_0^1 \frac{e^{s^2-1}}{20}\cos u(s)ds, \\ {}^{c}D_q^{\frac{5}{3}}u(0) + {}^{c}D_q^{\frac{5}{3}}u(\frac{3}{4}) + {}^{c}D_q^{\frac{5}{3}}u(1) = \int_0^1 \frac{2s^3+1}{20\pi}\cos u(s)ds, \end{array} \tag{3.5} $$

where $t \in J = [0,1]$, $\alpha = \frac{9}{4}$, $\beta = \frac{2}{3}$, $\gamma = \frac{5}{3}$ and $p = \frac{3}{4}$ in equations (1.1) and (1.2). We define maps $T : J \times \mathbb{R}^3 \to P(\mathbb{R})$ by $T(t, x_1, x_2, x_3) = [0, \frac{t^2}{100}\sin x_1 + \frac{1}{100}\cos x_2 + \frac{1}{100}(\frac{|x_3|}{1+|x_3|})]$, Also, $f_i : J \times \mathbb{R} \to \mathbb{R}$ define by $f_0(t,x) = \frac{t^2}{20}\cos x$, $f_1(t,x) = \frac{e^{t^2-1}}{20}\cos x$, $f_2(t,x) = \frac{2t^3+1}{300\pi}\cos x$, and $N : C^2(J) \to 2^{C^2(J)}$ by $N(u) = \{h \in C^2(J) \,|\, \exists v \in S_{T,u} : h(t) = w(t)\}$, for all t $\in J$ such that $w(t) = I_q^{\frac{9}{4}}v(t) + \frac{1}{3}\int_0^1 \frac{s^2}{20}\cos u(s)ds - \frac{1}{3}[I_q^{\frac{9}{4}}v(1) + I_q^{\frac{9}{4}}v(\frac{3}{4})] + a_1(t)\int_0^1 \frac{e^{s^2-1}}{20}\cos u(s)ds + a_2(t)[I_q^{\frac{19}{12}}v(1) + I_q^{\frac{19}{12}}v(\frac{3}{4})] + (b_1 + a_3(t))\int_0^1 \frac{2s^3+1}{300\pi}\cos u(s)ds + (b_2 + a_4(t))[I_q^{\frac{7}{12}}v(1) + I_q^{\frac{7}{12}}v(\frac{3}{4})]$. where $a_1(t)$, $a_2(t)$, $a_3(t)$, $a_4(t)$, b_1 and b_2 are calculated by (3.3). Put $m(t) = \frac{3t}{20}$, $m_0(t) = \frac{t^2}{20}$, $m_1(t) = \frac{e^{t^2-1}}{20}$, $m_2(t) = \frac{2t^3+1}{300\pi}$ and $\psi(t) = \frac{t}{5}$. In accordance with data of Table (1) In the article source file, it is easy to check that $H_d(T(t,u_1,u_2,u_3), F(t,v_1,v_2,v_3)) \le (m(t)\theta(\sum_{k=1}^{3}|u_k - v_k|))/(\Lambda_1+\Lambda_2+\Lambda_3)$, and $|f_j(t,u) - f_j(t,v)| \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}m_j(t)\psi(|u-v|)$, for $t \in J$, $j = 0,1,2$. Because $\sup_{u\in N(0)}\|u\| = 0$, we have $\inf_{u\in C^2(J)}\left(\sup_{v\in N(u)}\|u-v\|\right) = 0$. Thus, N has the approximate endpoint property. At present, by applying Theorem 3.2, the system of fractional q-differential inclusions (3.4) and (3.5) has at least one solution.

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