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SPECTRAL COLLOCATION METHODS FOR SOLVING FRACTIONAL OPTIMAL CONTROL PROBLEMS OF VARIABLE ORDER

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Abstract. In this paper, we consider the fractional optimal control problems (FOCPs) with variable fractional order derivatives. The time and space derivatives are replaced by the variable-order Caputo fractional derivatives. An excellent numerical method based on the spectral collocation method for the solutions are proposed. The proposed method is an indirect approach since we act on the equivalent Volterra integral equation of the second kind. Unconditional stability and a special case convergence of the proposed method are proved by the mathematical induction. Some numerical examples are given to support our theoretical analysis.

1. Introduction

Optimal control theory is a branch of optimization theory focused on minimizing a cost or maximizing a payoff. The fractional optimal control theory is a relatively new area in mathematics and engineering disciplines. FOCPs can be defined using different definitions of fractional derivatives, like the Riemann-Liouville and Caputo fractional derivatives as the most important ones. The authors in [1] have investigated the necessary conditions for optimization of FOCPs. Since the order of fractional derivatives and integrals may take any arbitrary value, another extension is considering the order not to be constant. This provides an extension of the classical fractional calculus, namely variable-order fractional calculus. Recently, several researchers have investigated and shown that many complex physical models can be described via variable order fractional derivatives with a great success. In a similar manner, variable order fractional optimal control problems (VO-FOCPs) can be defined with respect to different definitions of variable order fractional derivatives like the

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Riemann-Liouville and Caputo derivatives as the most important ones. Some recent works on VO-FOCPs can be found in [2, 3].

In this paper, we solve this problem by introducing and analyzing a Jacobi spectral collocation method for the following VO-FOCP:

$$min\ J(u) = \int_0^1 F(t, x(t), u(t))dt, \tag{1.1}$$

with the variable order fractional dynamical system:

$${}_0^C D_t^{\alpha(t)} x(t) = G(t, x(t), u(t)), \ \ 0 < \alpha(t) \leq 1, \ t \in [0, 1], \tag{1.2}$$

and the initial condition:

$$x(0) = x_0, \tag{1.3}$$

where F and G are smooth functions and ${}_0^C D_t^{\alpha(t)}$ denotes the variable order fractional derivative of $x(t)$ of order $\alpha(t)$ in the Caputo sense, which will be given through the following definitions.

Definition 1.1. The Riemann-Liouville variable order fractional integral operator of order $\alpha(t) \geq 0$ of a function $f(t)$ defined by:

$${}_{t_0}I_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_{t_0}^t (t-s)^{\alpha(t)-1} f(s) ds \tag{1.4}$$

Definition 1.2. The left and right Riemann-Liouville variable order fractional derivatives of $f(t)$ for $n-1 < \alpha(t) \leq n$ are as follows:

$${}_{t_0}^{RL} D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{n-\alpha(t)-1} f(s) ds, \tag{1.5}$$

and

$${}_{t}^{RL} D_{t_f}^{\alpha(t)} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_{t}^{t_f} (s-t)^{n-\alpha(t)-1} f(s) ds, \tag{1.6}$$

respectively.

Definition 1.3. The left and right Caputo variable order fractional derivatives of $f(t)$ for $n-1 < \alpha(t) \leq n$ are defined respectively by:

$${}_{t_0}^{C} D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_{t_0}^t (t-s)^{n-\alpha(t)-1} f^{(n)}(s) ds, \tag{1.7}$$

and

$${}_{t}^{C} D_{t_f}^{\alpha(t)} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \int_{t}^{t_f} (s-t)^{n-\alpha(t)-1} f^{(n)}(s) ds. \tag{1.8}$$

The proposed method consists of reducing the VO-FOCP in Eq.(1.1) and the VO-fractional dynamical system in Eqs. (1.2)-(1.3) to a system of nonlinear Volterra integral quations which can be simply solved. To this end, in the next section, this VO-FOCP is transformed into an equivalent system of Volterra integral equations. In Section 3, the spectral Jacobi collocation method is described. Extensive numerical experiments are provided in Section 4 to confirm our theoretical results.

2. Setting the problem

The first-order optimality conditions of VO-FOCP (1.1)-(1.3) are:

$$ {}_0^C D_t^{\alpha(t)} x(t) = \partial_\lambda H(t, x(t), u(t), \lambda(t)), \; x(0) = x_0 \tag{2.1} $$

$$ {}_t^{RL} D_1^{\alpha(t)} \lambda(t) = \partial_x H(t, x(t), u(t), \lambda(t)), \; \lambda(0) = 0 \tag{2.2} $$

$$ 0 = \partial_u H(t, x(t), u(t), \lambda(t)), \tag{2.3} $$

where $H(t, x(t), u(t), \lambda(t)) = F(t, x(t), u(t)) + \lambda(t)G(t, x(t), u(t))$ is the Hamiltonian function and $\lambda(.)$ is a Lagrange multiplier vector. By computing $u(t)$ from $\partial_u H(t, x(t), u(t), \lambda(t)) = 0$ and applying the fundamental theorem of fractional calculus, we get the following equivalent Volterra integral equations:

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha(t))} \int_0^t \partial_\lambda H(\tau, x(\tau), \lambda(\tau))(t-\tau)^{\alpha(t)-1} d\tau, \tag{2.4} $$

$$ \lambda(t) = \frac{1}{\Gamma(\alpha(t))} \int_t^1 \partial_x H(\tau, x(\tau), \lambda(\tau))(\tau - t)^{\alpha(t)-1} d\tau. $$

Theorem 2.1. *Let the function H be continuous and satisfies a Lipschitz condition with respect to the second variable with some constant $L > 0$. Then, the Volterra Eqs. (2.4) possesses a uniquely determined solution $(x, \lambda) \in C^1[0, 1]$ [4].*

3. The Jacobi spectral collocation scheme

In this section, the above firstorderoptimalitycondition is approximated byusinga fractional spectral collocation method based on the Jacobi polynomials. For the state Eq.(2.4), we approximate the state $x(t)$ and costate $\lambda(t)$ by

$$ x(t) \cong x_N(t) = \sum_{i=0}^{N} x_i J_i(t), \qquad \lambda(t) \cong \lambda_N(t) = \sum_{j=0}^{N} \lambda_j J_j(t), \tag{3.1} $$

where $J_p(t)$, $p = 0, 1, \cdots, N$, are the fractional Jacobi polynomials of degree p which are orthogonal polynomials with the weight $(1-t)^\delta t^\gamma$, $\delta, \gamma > -1$ [5]. Also, the derivative of fractional Jacobi polynomials are orthogonal fractional polynomials. The above expansion can also be alternatively expressed as

$$ x_N(t) = \sum_{i=0}^{N} x_N(t_i) J_i(t), \qquad \lambda_N(t) = \sum_{j=0}^{N} \lambda_N(t_j) J_j(t), \tag{3.2} $$

where t_p, $p = 0, 1, \cdots, N$, are interpolation points that we consider Legendre-Gauss points as the interpolation points. Then by (3.2) and (2.4), a direct computation leads to:

$$x_N(t) = x_0 + \frac{1}{\Gamma(\alpha(t))} \int_0^t \partial_\lambda H(\tau, x_N(\tau), \lambda_N(\tau))(t-\tau)^{\alpha(t)-1} d\tau, \tag{3.3}$$

$$\lambda_N(t) = \frac{1}{\Gamma(\alpha(t))} \int_t^1 \partial_x H(\tau, x_N(\tau), \lambda_N(\tau))(\tau-t)^{\alpha(t)-1} d\tau. \tag{3.4}$$

The system (3.3) can be solved by an iterative process (e.g., the Newton-Raphson iteration method or the successive substitution method).

4. Numerical Example

Consider the following VO-FOCP:

$$J(u) = \frac{1}{2} \int_0^1 \Big(x^2(t) + u^2(t)\Big) dt,$$

$${}_0^C D_t^{\alpha(t)} = -x(t) + u(t), \quad 0 < \alpha(t) \leq 1,$$

with $x(0) = 1$. To validate the accuracy of the achieved results, see Table 1.

Table 1. Optimal cost for the estimated value of J at different choices of $\alpha(t)$

$\alpha(t)$	[5]	[6]	$N = 100$	$N = 200$	$N = 400$
$1 - 0.2t$	0.16711	0.167347	0.167127	0.167119	0.167112
$1 - 0.1t$	0.17953	0.179690	0.179548	0.179540	0.179534
1	0.192909	0.192909	0.192909	0.192909	0.192909

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