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A PSEUDOSPECTRAL METHOD FOR SOLVING VARIABLE ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Variable-order differential operators can be employed as a powerful tool to modeling fractional differential equations (FDEs) and chaotical systems. In this paper, we consider the FDEs with variable fractional order derivatives. The time and space derivatives are replaced by the variable-order Caputo fractional derivatives. An excellent numerical method based on the spectral method for the solutions are proposed. The numerical schemes are based on the fundamental theorem of fractional calculus and the Lagrange polynomial interpolation. The obtained numerical results are an indication of the behavior of the results.

1. Introduction

Fractional calculus allows the operators of integration and differentiation to have fractional order. Fractional-order differential equations can be used to model many physical systems with more accuracy in relation to the integer-order equations due to their non-local properties. Hence, they have been further discussed and applied in the last few decades. An interesting extension of the constant order fractional calculus was proposed named variable-order fractional calculus [3]. Since it is not possible to find exact solutions of variable-order fractional differential equations (VO-FDEs), the developing numerical schemes for solving these equations is an important area of inquiry. Recently, spectral methods have been applied to FDEs, offering the benefit of more natural nonlocal approximations in addition to high accuracy in the case of smooth solutions [2, 4, 5]. Also, the pseudospectral method is one of the most popular direct methods which has been applied by many researchers [6]. The aim of this paper is to use the pseudospectral method with special fractional powers in which the use of these functions is important because we can obtain accurate solutions

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with a few number of Lagrange functions. We contribute to these methods by introducing and analyzing a Lagrange functions for the following rational-order fractional initial value problem:

$$ {}_0^C D_t^{\alpha(t)} x(t) = f(t, x(t)),\ x(0) = x_0, \tag{1.1} $$

where f is a continuous function and ${}_0^C D_t^{\alpha(t)}$ is the Caputo fractional derivative of variable-order defined in the next.

The paper is organized as follows. Notations and basic definitions of variable-order fractional derivatives are given in Section 2. In Section 3 the novel variable-order numerical method with Lagrange functions is presented. Furthermore, some illustrative examples are given in Section 4.

2. Preliminaries

In this section, we provide basic definitions of variable-order fractional derivatives which are used in the subsequent sections [1].

Definition 2.1. The Riemann-Liouville variable order fractional integral operator of order $\alpha(t) \geq 0$ of a function $f(t)$ defined by:

$$ {}_{t_0}I_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_{t_0}^{t} (t-s)^{\alpha(t)-1} f(s) ds \tag{2.1} $$

Definition 2.2. The left and right Riemann-Liouville variable order fractional derivatives of $f(t)$ for $n-1 < \alpha(t) \leq n$ are as follows:

$$ {}_{t_0}^{RL} D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-\alpha(t)-1} f(s) ds, \tag{2.2} $$

and

$$ {}_{t}^{RL} D_{t_f}^{\alpha(t)} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_{t}^{t_f} (s-t)^{n-\alpha(t)-1} f(s) ds, \tag{2.3} $$

respectively.

Definition 2.3. The left and right Caputo variable order fractional derivatives of $f(t)$ for $n-1 < \alpha(t) \leq n$ are defined respectively by:

$$ {}_{t_0}^{C} D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_{t_0}^{t} (t-s)^{n-\alpha(t)-1} f^{(n)}(s) ds, \tag{2.4} $$

and

$$ {}_{t}^{C} D_{t_f}^{\alpha(t)} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \int_{t}^{t_f} (s-t)^{n-\alpha(t)-1} f^{(n)}(s) ds. \tag{2.5} $$

3. Formulation of Pseudospectral Method

Consider the set of $\{t_j\}_{j=0}^{N-1}$ of zeros of Jacobi polynomials $\{J_N^{a,b}\}$ with parameters $a = 0$, $b = -1 + \frac{1}{\alpha}$ on $[0, 1]$. The jth Lagrange polynomial, $L_j(t)$, of order $N - 1$ is defined by

$$L_j(t) := \prod_{i=0}^{N-1} \frac{t - t_i}{t_j - t_i}, \quad j = 0, 1, \cdots, N - 1.$$

We make the change of variable $t = \tau^\alpha$ and obtain the jth fractional power Lagrange function defined by

$$L_j^\alpha(t) := \prod_{i=0, i\neq j}^{N-1} \frac{\tau^\alpha - t_i}{t_j - t_i}, \quad j = 0, 1, \cdots, N - 1.$$

It is worth noting that these functions preserve the orthogonal property of Lagrange polynomials. Then, we approximate the state variables $x(\tau)$ by fractional power Lagrange functions of degree at most N as follows:

$$x(\tau) \cong x_N(\tau) = \sum_{j=0}^{N-1} x_N(\tau_j) L_j^\alpha(\tau), \quad j = 0, 1, \cdots, N - 1. \tag{3.1}$$

Now, we obtain the fractional differentiation matrix at the collocation points $\{\tau_i\}_{i=0}^{N-1}$ as

$${}_0D_\tau^{\alpha(t)} x_N(\tau)|_{\tau=\tau_i} = \sum_{j=0}^{N-1} x_N(\tau_j) D_{ij}^{\alpha(\tau_i)} \tag{3.2}$$

where $D_{ij}^{\alpha(\tau_i)}$ denotes the ijth entry of the fractional differentiation matrix as follows:

$$D_{ij}^{\alpha(\tau_i)} = \frac{1}{t_j} \sum_{k=0}^{N-1} c_{kj} \sum_{m=0}^{k} d_{km} (\tau_i^{m\alpha(\tau_i))} \tag{3.3}$$

where

$$d_{km} = \frac{(-1)^{k-m} \Gamma(1 + b + k) \Gamma(1 + a + b + k + m) \Gamma(m\alpha_i + \alpha_i + 1)}{m!(k - m)! \Gamma(1 + b + m) \Gamma(1 + a + b + k) \Gamma(m\alpha_i + 1)}$$

$$c_{ij} = \frac{2\alpha_i + 1}{\alpha_i 2^{\frac{1}{\alpha_i}}} w_j L_i(\tau_j; \alpha_i)$$

in which $\alpha_i = \alpha(\tau_i)$ and $L_n(t; \alpha)$ expressed the Muntz-Legendre polynomials. In this framework, this scheme is directly considered to obtain a numerical solution of the system (1.1).

4. Numerical Example

Consider the following VO-FDE:

$$ {}_0^C D_t^{\alpha(t)} = -x(t) + \frac{2}{\Gamma(3-\alpha(t)} t^{2-\alpha(t)} - \frac{1}{\Gamma(2-\alpha(t)} t^{1-\alpha(t)} + t^2 - t, \quad t \in [0,1], $$

with $x(0) = 1$. The exact solution is $x(t) = t^2 - t$. To validate the accuracy of the achived results, see Table 1.

Table 1. Absolut errors at different values of $\alpha(t)$ and N

$\alpha(t)$	$N = 5$	$N = 7$	$N = 10$
$1 - 0.01t$	6.875×10^{-8}	8.580×10^{-11}	1.052×10^{-13}
0.95	3.657×10^{-11}	1.173×10^{-12}	4.319×10^{-14}
1	2.589×10^{-11}	2.639×10^{-13}	1.833×10^{-15}

5. Conclusions

A numerical scheme based on Lagrange polynomial interpolation and the fundamental theorem of fractional calculus was proposed to get a numerical solutions for VO-FDEs. The method is accurate, efficient and direct. Numerical examples with different variable-orders have been presented to demonstrate the effectiveness of the method.

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