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OPTIMAL MODEL PREDICTIVE STABILIZATION CONTROL OF A DOUBLE INVERTED PENDULUM ON A CART

FAHIMEH ALSADAT KAZEMI[1]*

[1] *Department of Applied Mathematics, Payame Nour University,*
P.O. Box 91735-433, Mashhad, Iran.
f.sadat.kazemi@gmail.com

Abstract. An optimal model predictive control system is proposed here. The controller is applied to stabilize a double inverted pendulum on a cart. The only control variable is a simple force applied to the cart, where, six variables including the positions and velocities of the cart, first link, and the second link of the pendulum must be stabilized. Thus, it is an underactuated control system. Here, the system's nonlinear equations of motion are studied and linearized around the equilibrium point. Then, they are discretized using Euler zero-order-hold method. The control signal is optimized based on the behavior of system during a moving forward window. The proposed control approach is applied to the system, and the results confirm efficiency of the proposed controller.

1. Introduction

In a variety of real applications such as vehicles, aerospace gadgets, robotic tools, etc., the system is governed by a fewer control input(s) compare to the outputs; such systems are termed underactuated systems. Designing a controller for an underactuated system is much more challenging and harder than for a fully actuated system [1]. Inverted pendulum is a famous benchmark of control designing process. However, when a double inverted pendulum is mounted over a wheeled cart, an underactuated, fast responding, nonlinear and hard-to-control system is shaped. Double Inverted Pendulum on a Cart (DIPOAC) is a good instance to evaluate the performance of controllers [2]. Many scientific works have been done for modeling and stabilizing the DIPOAC. The model-based Linear Quadratic Regulator (LQR) algorithm [3], intermittent predictive pole-placement control method [4], self-tuning Proportional-Integral-Derivative (PID) control strategy [5] and RBFARX-based

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* Speaker.

model predictive control approach [1], are some examples. A prosperous control method for such systems is Model Predictive Control (MPC). Where, the system future behavior is predicted, and a finite-time-domain optimization problem is defined. Moreover, at every single sampling time, based on the solution of optimization problem, in respect to the current measurement information, the first element of the optimal solution vector is considered as the control signal. This process will be repeated in the next sampling step [6].
In this paper, we designed an optimal predictive controller for a DIPOAC, which is presented in the following. The rest of paper is organized as follows: in section II, nonlinear, linear and discrete-time models of the DIPOAC is described. Section III illustrates the proposed control method. In section IV, the results are discussed and then in the last section the conclusions are clarified.

2. Double Inverted Pendulum on a Card Model

As conceptual schematic of DIPOAC is illustrated in Figure 1. In its equations of motion, several output variables are used that $\theta_0, \theta_1, \theta_2$ are cart position, lower pendulum angle, and upper pendulum angle, respectively, and $\theta_0', \theta_1', \theta_2'$ are cart velocity, angular velocity of the lower pendulum, and angular velocity of the upper pendulum, respectively. For simplicity we define $\theta = [\theta_0, \theta_1, \theta_2]^T$ and $\theta' = [\theta_0', \theta_1', \theta_2']^T$. Therefore the dynamical model of the DIPOAC can be written as (2.1) [1].

$$D(\theta)\theta'' + C(\theta, \theta')\theta' + G(\theta) = Hu \tag{2.1}$$

in which,

$$D(\theta) = \begin{bmatrix} d_1 & d_2 cos\theta_1 & d_3 cos\theta_2 \\ d_2 cos\theta_1 & d_4 & d_5 cos(\theta_1 - \theta_2) \\ d_3 cos\theta_2 & d_5 cos(\theta_1 - \theta_2) & d_6 \end{bmatrix}, G(\theta) = \begin{bmatrix} 0 \\ -f_1 sin\theta_1 \\ -f_2 sin\theta_2 \end{bmatrix}$$

$$C(\theta, \theta') = \begin{bmatrix} 0 & -d_2 sin(\theta_1)\theta_1' & -d_3 sin(\theta_2)\theta_2' \\ 0 & 0 & d_5 sin(\theta_1 - \theta_2)\theta_2' \\ 0 & -d_5 sin(\theta_1 - \theta_2)\theta_1' & 0 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where, $d_1 = m_0 + m_1 + m_2, d_2 = (m_1/2 + m_2)L_1, d_3 = m_2 L_2/2, d_4 = (m_1/3 + m_2)L_1^2, d_5 = m_2 L_1 L_2/2, d_6 = m_2 L_2^2/3, f_1 = (m_1/2 + m_2)L_1 g, f_2 = m_2 L_2 g/2$.
The system can be linearized around the main equilibrium point, $(\theta_0, \theta_1, \theta_2, \theta_0', \theta_1', \theta_2') = (0, ..., 0)$, with the state vector $x = [\theta^T, \theta'^T]^T$ as (2.2).

$$\begin{aligned} x'(t) &= A_c x(t) + B_c u(t) \\ y(t) &= x(t) \end{aligned} \tag{2.2}$$

in which, $Ac = \begin{bmatrix} 0 & I \\ D(0)^{-}1\frac{dG(0)}{d\theta} & 0 \end{bmatrix}, Bc = \begin{bmatrix} 0 \\ D(0)^{-}1H \end{bmatrix}$. After discretizing the model using zero order hold with sampling period T = 0.002s, the discrete time model can be rewritten as (2.3)

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= x(k) \end{aligned} \tag{2.3}$$

in which, $A = \begin{bmatrix} 1 & 0 & 0 & 0.0020 & 0 & 0 \\ 0 & 1.0001 & -0.0001 & 0 & 0.0020 & 0 \\ 0 & -0.0001 & 1.0001 & 0 & 0 & 0.0020 \\ 0 & -0.0150 & -0.0016 & 1 & 0 & 0 \\ 0 & 0.1499 & -0.0674 & 0 & 1.0001 & -0.0001 \\ 0 & -0.1199 & 0.1042 & 0 & -0.0001 & 1.0001 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0012 \\ -0.0030 \\ 0.0006 \end{bmatrix}.$

3. Optimal Model Predictive Stabilization Controller

If the system output is considered in a finite-time forward moving window (with the length of N), then we have:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ x(k+2) &= Ax(k+1) + Bu(k+1) = A^2x(k) + ABu(k) + Bu(k+1) \\ &\vdots \\ x(k+N) &= A^N x(k) + \sum_{i=0}^{N-1} (A^{N-1-i} Bu(k+i)) \end{aligned} \tag{3.1}$$

Therefore, the output trajectory through the moving window is:

$$X_k = \Gamma x(k) + \Psi U_k \tag{3.2}$$

in which,

$$X_k = \begin{bmatrix} x(k+1) \\ \vdots \\ x(k+N) \end{bmatrix}, U_k = \begin{bmatrix} u(k) \\ \vdots \\ u(k+N-1) \end{bmatrix}, \Gamma = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \Psi = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

The cost-functional of optimization is defined here as:

$$J_{k,N} = trace(X_k^T X_k) = trace\{U_k^T \Psi^T \Psi U_k + 2x(k)^T \Gamma^T \Psi U_k + x(k)^T \Gamma^T \Gamma x(k)\} \tag{3.3}$$

where, the optimal control signal, obtained as a solution for $\frac{\partial J}{\partial u} = 0 \rightarrow \Psi^T \Psi U_k + \Psi^T \Gamma x(k) = 0$, and we have $U_k^* = \Phi x(k)$ in which, $\Phi = (\Psi^T \Psi)^{-1} \Psi^T \Gamma$, and $u(k)^*$ is the first element of U_k^*. Applying this optimal predictive control signal to the system, the system's next output will be obtained.

$$x^*(k+1) = Ax(k) + Bu^*(k) = (A + B\Phi_1)x(k) \tag{3.4}$$

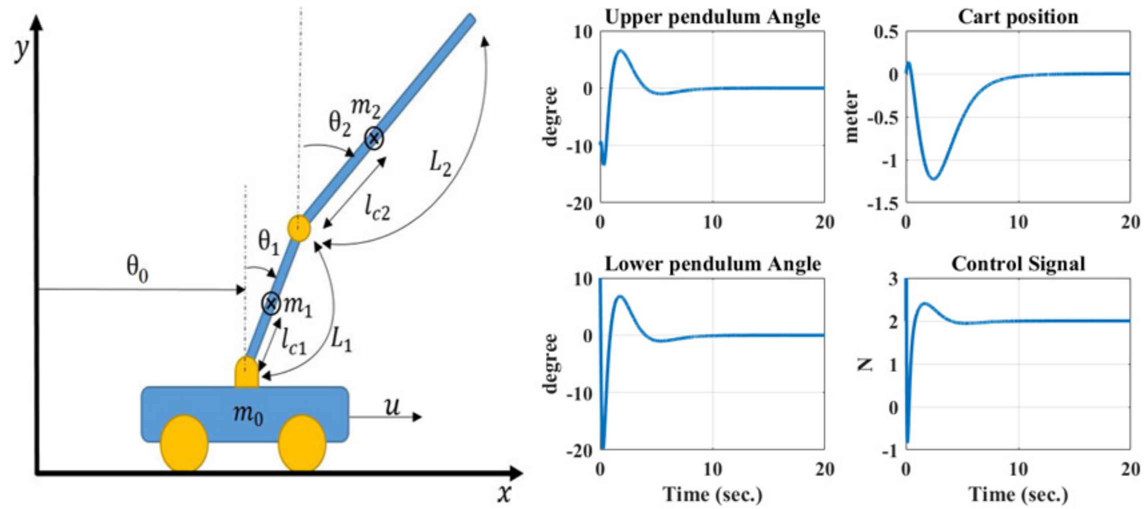

FIGURE 1. (Left) Conceptual Schematic of the DIPOAC, (Right) result of the simulations.

in which, Φ_1 is the first raw of Φ. Applying the proposed controller to control the DIPOAC system, we can see the output trajectory as it is seen in Figure 1.

4. Conclusion

An optimal predictive stabilization controller is proposed here, and applied to control a double inverted pendulum that is attached on a cart. Design of control law for such a system is hard and challenging. Efficiency of the proposed controller is confirmed through simulations.

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