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NASH EQUILIBRIA IN THE FRACTIONAL DIFFERENTIAL GAME

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Abstract. In this paper we consider Nash equilibria for the affine linear quadratic fractional differential game for a finite planning horizon where the dynamic system depends on Caputo fractional derivatives. The Nash equilibrium is a proposed solution of a noncooperative game involving two or more players in which each player assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. According to the Pontryagin minimum principle for optimal control problems and by constructing an error function, we define an unconstrained minimization problem. In order to solve this problem, we can use any optimization algorithms.

Keywords: Nash Equilibrium, Linear quadratic fractional differential games, Fractional optimal control problems, Caputo fractional derivative, Optimization.

1. Introduction

Game theory is the study of mathematical models of strategic interaction between rational decision-makers.[1] It has applications in all fields of social science, as well as in logic and computer science. The Nash equilibrium is a proposed solution of a noncooperative game involving two or more players in which each player assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy [2].

Fractional calculus, which has a lot of applications in science, mathematics and engineering (see [3]), can be considered as an extension of classical calculus. The main contributions in fractional calculus were made in last two decades where it could be applied to engineering and optimal control problems.

* Speaker.

2. Some preliminaries

2.1. **Fractional derivative.** Several definitions of a fractional derivative have been proposed [4]. The left Caputo fractional derivative (CFD), for a real function y, can be defined as follows:

$$ {}_a^cD_t^\alpha(y(t)) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}\Big(\frac{d}{d\tau}\Big)^n y(\tau)d\tau, \qquad n-1<\alpha<n. \tag{2.1}$$

2.2. **Properties of Legendre basis.** The analytic form of the shifted Legendre polynomials $p_n(t)$ of degree n on the interval $[0,1]$ is given by $p_n(t) = \sum_{m=0}^{n}(-1)^{n+m}\frac{(n+m)!t^m}{(n-m)!(m!)^2}$. The function $y(t)$ which belongs to the space of square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$ y(t) = \sum_{m=0}^{\infty} c_m p_m(t), \qquad c_m = (2m+1)\int_0^1 y(t)p_m(t)\mathrm{d}t, \qquad m = 0,1,\ldots. $$

3. Problem statement

In this section we introduce the problem. Let $\bar{N} = 1,2,\ldots,N$ be the set of players. The dynamic environment where the players interact is modeled by a linear time invariant fractional differential equation as

$$ {}_0^cD_t^\alpha(x(t)) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{3.1}$$

where $u_i(t) \in \mathbb{R}^{l_i}$, $A \in \mathbb{R}^{n\times n}$, and $B_i \in \mathbb{R}^{n\times l_i}$. Player i affects the state, $x(t)$, of the system (3.1) by choosing a strategy/control $u_i(t)$ at time $t \geq 0$. We assume that the control trajectory, denoted by u_i, belongs to an admissible control space U_i. We denote the joint control $u = (u_1, u_2, ..., u_N) \in U^1 \times U^2 \times \cdots \times U^N = U$ and $B = [B_1B_2 \ldots B_N] \in \mathbb{R}^{n\times l}$. Formally, each player i is assumed to minimize the cost

$$ J_i(u_i, u_{-i}) = \int_0^1 \frac{1}{2}\Big[x^T(t)Q_i x(t) + \sum_{j=1}^{N} u_j^T(t)R_{ij}u_j(t)\Big]\,\mathrm{d}t, \tag{3.2}$$

Note that the functions $g(x, u_1, \ldots, u_N) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t)$ and $f_i(t, x, u_1, \ldots, u_N) = \frac{1}{2}\Big[x^T(t)Q_i x(t) + \sum_{j=1}^{N} u_j^T(t)R_{ij}u_j(t)\Big]$ for $i = 1,\ldots,N$ where Q_i is symmetric positive semi definite and R_{ij}, $i,j = 1,\ldots,N$ are symmetric positive definite.

In performance index $J_i(u_i, u_{-i})$ is given in the (3.2), u_i is the control of i_{th} Player and u_{-i} are the controls for the rest of the players $u_{-i} = (u_j, j \neq i)$. For each player, the goal of game is to minimize (maximize) of his own performance index by selecting appropriate control function.

Definition 3.1. (Nash Equilibrium [2]). The players control actions $u_i^*(.)(i = 1,\ldots,N)$ are said to be in a Nash equilibrium if for any other admissible control actions $u_i(.)(i = 1,\ldots,N)$ the following inequalities hold: $J_i(u_i^*, u_{-i}^*) \leq J_i(u_i, u_{-i}^*), \quad \forall u_i \in U_i$.

(A): Assume that the dynamics and the running costs take the decoupled form

$$\begin{cases} f_i(x, u_1, \ldots, u_N) = \sum_{j=1}^{N} L_{ij}(t, x, u_j), \quad i = 1, \ldots, N \\ g(x, u_1, \ldots, u_N) = g_0(t, x) + \sum_{i=1}^{N} h_i(t, x) u_i. \end{cases}$$

Also, assume that $U_i (i = 1, \ldots, N)$ are closed and convex subsets of $\mathbb{R}^{l_i}$, the functions $u_i \mapsto L_{ii}(t, x, u_i)$ are strictly convex and for each $i = 1, \ldots, N$, either U_i is compact or L_{ii} has superlinear growth, $\lim_{|u_i| \to \infty} \frac{L_{ii}(t,x,u_i)}{u_i} = +\infty, \;\; i = 1, \ldots, N$. Then for every $(t, x) \in [0, T] \times \mathbb{R}^n$ and any vector $p_i(t) \in \mathbb{R}^n$, there exists a unique set $(u_1^*(t), \ldots, u_n^*(t)) \in U^1 \times U^2 \times \cdots \times U^N$ such that $u_i^*(t) = arg \;\; \min_{u_i \in U_i} p_i(t).h_i(t, x)u_i(t) + L_{ii}(t, x, u_i)(t)$. The assumptions in (A) guarantee that the above minimizers exist and are unique [2].

4. AN APPROXIMATION METHOD

In this section, a numerical scheme is presented to solve a Nash equilibria in the fractional game where the fractional derivative is defined in the Caputo sense. We consider problem (3.1). Hamiltonian functions for this problem is given by $H_i = f_i(t, x(t), u_1(t), \ldots, u_N(t)) + \lambda_i^T g(x(t), u_1(t), \ldots, u_N(t))$ where $\lambda_i \in \mathbb{R}^n$ are the Lagrange multipliers. The necessary conditions is given as

$$\begin{aligned} &{}_0^c D_t^\alpha x(t) = g(x(t), u_1(t), \ldots, u_N(t)), \quad x(0) = x_0, \\ &{}_t^c D_1^\alpha \lambda_i(t) = \frac{\partial H_i}{\partial x}(t, x(t), u_i(t), u_{-i}(t), \lambda_i(t)), \quad \lambda_i(1) = 0, \quad i = 1, \ldots, N, \\ &\frac{\partial H_i}{\partial u_i}(t, x(t), u_i(t), u_{-i}(t), \lambda_i(t)) = 0, \quad i = 1, \ldots, N, \end{aligned} \tag{4.1}$$

According to the PMP, if $(x(t)^T, u_i(t)^T)^T$ be an optimal solution of problem (3.1), then there are $\lambda_i(t)$ such that $(x(t)^T, u_i(t)^T)^T$ satisfies in (4.1). The basic idea of these methods is to expand the solution function as a finite series of very smooth basis functions, as given

$$x(t) \simeq x^s(t) = \sum_{m=0}^{s} c_m^x p_m(t), \quad u_i(t) \simeq u_i^s(t) = \sum_{m=0}^{s} c_m^{u_i} p_m(t), \quad \lambda_i(t) \simeq \lambda_i^s(t) = \sum_{m=0}^{s} c_m^{\lambda_i} p_m(t),$$

where $p_m(t)$ are shifted Legendre polynomials. We propose shifted Legendre polynomials to estimate control, state and co-state functions which this trial solutions satisfy the initial or boundary conditions as

$$\bar{x^s}(t) = D(t) + F(t) x^s(t), \quad \bar{u_i^s}(t) = J(t) + K(t) u_i^s, \quad \bar{\lambda_i^s}(t) = L(t) + O(t) \lambda_i^s, \tag{4.2}$$

where $D(t)$, $F(t)$, $J(t)$, $K(t)$, $L(t)$ and $O(t)$ are real single variable functions such that the approximations of $\bar{x^s}$, $\bar{u_i^s}$ and $\bar{\lambda_i^s}$ satisfy the initial or final conditions in (4.1). The trial solutions (4.2) are the universal approximation and must satisfy conditions (4.1). Thus, we have

$$ {}_0^c D_t^\alpha \bar{x^s}(t) = g(\bar{x^s}(t), \bar{u_1^s}(t), \ldots, \bar{u_N^s}(t)), \qquad {}_t^c D_1^\alpha \bar{\lambda_i^s}(t) = \frac{\partial \bar{H}_i}{\partial \bar{x^s}}, \qquad \frac{\partial \bar{H}_i}{\partial \bar{u_i^s}} = 0, \tag{4.3}$$

where $\bar{H}_i = H(t, \bar{x^s}(t), \bar{u^s_i}(t), \bar{u^s}_{-i}(t))$ for $i = 1, \ldots, N$. In order to reformulate (4.3) as an unconstrained minimization problem, we first collocate the optimality system (4.3) on the s points. We choose the grid points to be the Chebyshev-Gauss-Lobatto points (see [5]) associated with the interval $[0, 1]$ as $t_k = \frac{1}{2}(1 - \cos(\frac{\pi k}{s})), \quad k = 1, 2, \ldots, s$. For the discretized time interval $[0, 1]$ with s grid points, left and right fractional derivatives (2.1) can be approximately written as follows:

$$ {}_0^c D_t^\alpha(\bar{x^s}(t_k)) = \frac{1}{\Gamma(n-\alpha)} \int_0^{t_k} (t_k - \tau)^{n-\alpha-1} \Big(\frac{d}{d\tau}\Big)^n \ \bar{x^s_i}(\tau) d\tau, \qquad n-1 < \alpha < n, $$

$$ {}_t^c D_1^\alpha(\bar{\lambda}^s_i(t_k)) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{t_k}^{1} (\tau - t_k)^{n-\alpha-1} \Big(\frac{d}{d\tau}\Big)^n \ \bar{\lambda}^s_i(\tau) d\tau, \qquad n-1 < \alpha < n. $$

Hence, we define an optimization problem as

$$ E(\Omega) = \frac{1}{2} \sum_{k=1}^{s} \sum_{i=1}^{N} \big\{ E_1(t_k, \Omega) + E_2^i(t_k, \Omega) + E_3^i(t_k, \Omega) \big\}, \tag{4.4} $$

where $\Omega = (C^x, C^{u_1}, \ldots, C^{u_N}, C^{\lambda_1}, \ldots, C^{\lambda_N})^T$, $C^x = (c_0^x, \ldots, c_s^x)$, $C^{u_i} = (c_0^{u_i}, \ldots, c_s^{u_i})$ and $C^{\lambda_i} = (c_0^{\lambda_i}, \ldots, c_s^{\lambda_i})$ and E_1, E_2^i and E_3^i are sum of the squares of the sides of (4.3) equations.

Lemma 4.1. *If* $\Omega^* = (C^{*x}, C^{*u_1}, \ldots, C^{*u_N}, C^{*\lambda_1}, \ldots, C^{*\lambda_N})$ *satisfies the following equation*

$$ \eta(\Omega) = \big[E_1(t_1, \Omega), \ldots, E_1(t_s, \Omega), E_2^1(t_1, \Omega), \ldots, E_2^1(t_s, \Omega), \ldots, E_2^N(t_1, \Omega), \\ \ldots, E_2^N(t_s, \Omega), E_3^1(t_1, \Omega), \ldots, E_3^1(t_s, \Omega), \ldots, E_3^N(t_1, \Omega), \ldots, E_3^N(t_s, \Omega) \big]^T = 0, $$

then Ω^* *is an optimal solution of* (4.4). **Proof.** *See* [6].

In order to solve (4.4), which is an unconstrained optimization problem, we can use any optimization algorithms.

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