A NEW ATTITUDE TO CONTROLLABILITY AND OBSERVABILITY OF DICCRETE TIME LINEAR SYSTEMS WITH INTERVAL COEFFICIENTS

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Abstract. In this paper, the controllability and observability of discrete time linear systems with interval coefficients by using of full rank interval matrix, are investigated. The full rank interval matrix will be defined and the being full rank of interval matrix is expressed by means of its definition or some criteria.

1. Introduction

At first, the interval parameter transformation is expressed. Consider the closed bounded interval $\mathsf{S} = [s^0, s^1] = \{x \in \mathbb{R} | s^0 \leqslant x \leqslant s^1\}$ for $s^0, s^1 \in \mathbb{R}$. Any value in S may be stated as $s(\lambda) = s^0 + \lambda(s^1 - s^0), 0 \leqslant \lambda \leqslant 1,$ and so the left and right endpoints of the interval $\mathsf{S} = [s^0, s^1]$ express as $s^0 = min\ s(\lambda), s^1 = max\ s(\lambda)\ ,(0 \leqslant \lambda \leqslant 1)$.[?]. $\mathsf{Y} = [y^0, y^1] = \{y(\lambda_2) | \lambda_2 \in [0, 1]\}$ is also a closed bounded interval.The algebraic operations of intervals are presented with respect to parameters as follows [?]:
$\mathsf{S} \oplus \mathsf{Y} = \{s(\lambda_1) + y(\lambda_2) | \lambda_1, \lambda_2 \in [0, 1]\}$,
$\mathsf{S} \ominus \mathsf{Y} = \{s(\lambda_1) - y(\lambda_2) | \lambda_1, \lambda_2 \in [0, 1]\}$,
$\mathsf{S} \odot \mathsf{Y} = \{s(\lambda_1).y(\lambda_2) | \lambda_1, \lambda_2 \in [0, 1]\}$,
$k\mathsf{S} = \{ks(\lambda) | \lambda \in [0, 1]\}$
$\mathsf{S} \oslash \mathsf{Y} = \{s(\lambda_1)/y(\lambda_2) | \lambda_1, \lambda_2 \in [0, 1], y(\lambda_2) \not\vDash 0\}$.
$\mathcal{S}$ is called an interval matrix if and only if all of its element are real numbers interval. Let $I(\mathbb{R})$ be the set of all closed intervals in $\mathbb{R}$, $I(\mathbb{R})^n$ be the product space $I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$, $I(\mathbb{R})^{m\times n}$ be the set of all interval matrices $\mathcal{S}$ with m rows and n colums and $J[0, 1]^{m\times n}$ be the set of all real matrices with m rows and n colums such that all

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elements of these matrices belong to $[0,1]$. Whenever $S=[s_{ij}]_{m\times n}$ is a real matrix and $\mathcal{S}=[\mathsf{S}_{ij}]_{m\times n}\in I(\mathbb{R})^{m\times n}$ is an interval matrix, where $\mathsf{S}_{ij}=[s^0_{ij},s^1_{ij}]$. $S\in\mathcal{S}$ if and only if $s_{ij}\in\mathsf{S}_{ij}$, for all $i=1,\dots,m$ and $j=1,\dots,n$.

Proposition 1.1. *An interval matrix $\mathcal{S}$ is presented by the infinite set of real matrices, i.e.*
$\mathcal{S}=\{S_\Lambda | S_\Lambda=[s_{ij}(\lambda_{ij})]_{m\times n}\ ,\ \Lambda=[\lambda_{ij}]_{m\times n}\in J[0,1]^{m\times n}\ ,\ s_{ij}(\lambda_{ij})=s^0_{ij}+\lambda_{ij}(s^1_{ij}-s^0_{ij})\ ,\ i=1,\dots,m\ ,\ j=1,\dots,n\}$

Proof. Suppose $S\in\mathcal{S}$ that, $S=[s_{ij}]_{m\times n}$, then $s_{ij}\in\mathsf{S}_{ij}\ ,\ i=1,\dots,m\ ,\ j=1,\dots,n$. Then, there exists $\lambda_{ij}\in[0,1]$ such that $s_{ij}=s^0_{ij}+\lambda_{ij}(s^1_{ij}-s^0_{ij})$. In this case, for each element of the matrix S, there exists a real number $\lambda_{ij}\in[0,1]\ ,\ i\in\{1,\dots,m\}\ ,\ j\in\{1,\dots,n\}$. Consider the real matrix $\Lambda=[\lambda_{ij}]_{m\times n}\ ,\ \lambda_{ij}\in[0,1]\ ,$ and suppose $s_{ij}=s^0_{ij}+\lambda_{ij}(s^1_{ij}-s^0_{ij})=s_{ij}(\lambda_{ij})$,for $i=1,\dots,m\ ,\ j=1,\dots,n$. Then,there is a real matrix $\Lambda=[\lambda_{ij}]\in J[0,1]^{m\times n}$, that $S=S_\Lambda$. Therefore, the real matrix S is a member of the infinite set of real matrices.
Now if a real matrix $S=[s_{ij}]_{m\times n}$ belongs to the infinite set of real matrices, then, there is a real matrix $\Lambda=[\lambda_{ij}]_{m\times n}\ ,\ \lambda_{ij}\in[0,1]$ such that $S=S_\Lambda$ and $s_{ij}=s_{ij}(\lambda_{ij})=s^0_{ij}+\lambda_{ij}(s^1_{ij}-s^0_{ij})\ ,\ \lambda_{ij}\in[0,1]\ ,\ i=1,\dots,m\ ,\ j=1,\dots,n$. Then $s_{ij}\in\mathsf{S}_{ij}\ ,\ i=1,\dots,m\ ,\ j=1,\dots,n$, and so $S\in\mathcal{S}$. □

The most important advantage of defining an interval matrix in the above proposition is that it shows all real matrices which are included within interval matrix. An interval matrix $\mathcal{S}$ is called full rank if and only if for each matrix $\Lambda\in J[0,1]^{m\times n}$, the real matrix $S_\Lambda\in\mathcal{S}$ is full rank.
Let $S=[s_{ij}]_{m\times n}$ and $Y=[y_{ij}]_{m\times n}$ are real matrices. Absolute of matrix is denoted by $|S|=[|s_{ij}|]_{m\times n}$ and also $S\leqslant Y$ if and only if $s_{ij}\leqslant y_{ij}$ for $i=1,\dots,m\ ,\ j=1,\dots,n$. The pseudoinverse of S, is a real $n\times m$ matrix as S^+ such that SS^+ and S^+S are symmetric matrices and $SS^+S=S\ ,\ S^+SS^+=S^+$. Suppose that, S is full rank and m is greater than or equal to n, then $S^+=(S^TS)^{-1}S^T$ and S^+S is a unit $n\times n$ matrix, otherwise, m is smaller than or equal to n, then $S^+=S^T(SS^T)^{-1}$ and SS^+ is a unit $m\times m$ matrix [**?**]. The singular values of the matrix S are as arithmetic square roots of the joint eigenvalues of the matrices S^TS and SS^T. $\sigma_{min}(S)$ and $\sigma_{max}(S)$ are respectively the smallest and greatest singular values of the matrix S. $\rho(\cdot)$ is the spectral radius of a square matrix.
Let $\mathsf{S}=[s^0,s^1]$ be an interval of real numbers. The midpoint of the interval, the radius of the interval and the absolute of the interval denote by, $mid\ \mathsf{S}=\frac{s^0+s^1}{2}\ ,\ rad\ \mathsf{S}=\frac{s^1-s^0}{2}\ ,\ |\mathsf{S}|=\max\{|s^0|\ ,\ |s^1|\}$ respectively. The operations of an interval as, midpoint, radius and absolute value are elementwisely extended for interval matrices. The matrices $\mathcal{S}_{mid}$ and $\mathcal{S}_{rad}$ are respectively called the midpoint matrix and the radius matrix of an interval matrix $\mathcal{S}$.

Theorem 1.2. *An interval $m\times n$ matrix $\mathcal{S}\ ,\ m\geqslant n$, is full rank if and only if the system of inequalities $|\mathcal{S}_{mid}x|\leqslant\mathcal{S}_{rad}|x|\ ,\ x\in\mathbb{R}^n$, has a unique zero solution.*

Proof. This theorem is proved in [3]. □

By using of (??), some criteria are proposed as following that being full rank of the interval matrices is investigated by them.If the matrix $\mathcal{S}_{mid}$ is full rank and $\rho(|(\mathcal{S}_{mid})^{+}| \cdot \mathcal{S}_{rad}) < 1$ then the interval matrix $\mathcal{S}$ is full rank by spectral radius criterion. If the inequality $\sigma_{max}(\mathcal{S}_{rad}) < \sigma_{min}(\mathcal{S}_{mid})$, is satisfied the interval matrix $\mathcal{S}$ is full rank by singular value criterion. If the matrix $\mathcal{S}_{mid}$ is full rank and the inequality $\| \mathcal{S}_{rad} \| < \| (\mathcal{S}_{mid})^{+} \|^{-1}$ is satisfied, the interval matrix $\mathcal{S}$ is full rank by norm criterion. $\| \bullet \|$ is an absolute subordinate matrix norm.

2. Main results

Consider the discrete time linear system with interval coefficients

$$x_{k+1} = \mathcal{A}x_k + \mathcal{B}u_k \qquad\qquad y_k = \mathcal{C}x_k + \mathcal{D}u_k \tag{2.1}$$

where $,\mathcal{A} \in I(\mathbb{R})^{n\times n}$ $,\mathcal{B} \in I(\mathbb{R})^{n\times m}$ $,\mathcal{C} \in I(R)^{p\times n}$ $,\mathcal{D} \in I(R)^{p\times m}$, are interval matrices that they are defined according to (??). From the system (??), a system with real number coefficients gaines as follows:

$$x_{k+1} = A_{\Lambda}x_k + B_{\Gamma}u_k \qquad\qquad y_k = C_{\Theta}x_k + D_{\Omega}u_k \tag{2.2}$$

such that $A_\Lambda \in \mathcal{A}$, $B_\Gamma \in \mathcal{B}$, $C_\Theta \in \mathcal{C}$, and $D_\Omega \in \mathcal{D}$. The state-transition equation of (??) by the initial state x_0 is written as follows[?]:

$x_k = A_\Lambda^k x_l + \sum\limits_{i=1}^{k} A_\Lambda^{k-i} B u_{i-1}$

Definition 2.1. The linear system with interval coefficients by state-space equations in (??) is controllable if and only if the linear system by state-space equations in (??) is controllable for all matrices $\Lambda \in J[0,1]^{n\times n}$, $\Gamma \in J[0,1]^{n\times m}$, $\Theta \in J[0,1]^{p\times n}$, $\Omega \in J[0,1]^{p\times m}$.

Definition 2.2. The linear system by state-space equations in (??) is controllable whenever a certain control sequence $\{u_k\}$ can convey the state sequence $\{x_k\}$ from any initial position x_0 to any other position x_1 in a finite number of discrete time steps such that $x_0, x_1 \in \mathbb{R}^n$. In other words, the linear system by state-space equations in (??) is controllable if for any positions $x_0, x_1 \in \mathbb{R}^n$, there are $k \in \mathbb{N}$ and a sequence $\{u_k\}$ such that $A_\Lambda^k x_0 + \sum\limits_{i=1}^{k} A_\Lambda^{k-i} B u_{i-1} = x_1$.

Proposition 2.3. *The linear system with interval coefficients by state-space equations in (??) is controllable if and only if the interval compound matrix*

$$M_{\mathcal{AB}} = \begin{bmatrix} \mathcal{B} & \mathcal{AB} & \mathcal{A}^2\mathcal{B} & \dots & \mathcal{A}^{n-1}\mathcal{B} \end{bmatrix}$$

is full rank.

Proof. Let the system (**??**) is controllable and then for all matrices $\Lambda \in J[0,1]^{n\times n},\ \Gamma \in J[0,1]^{n\times m},\ \Theta \in J[0,1]^{p\times n},\ \Omega \in J[0,1]^{p\times m}$ the system (**??**) is controllable. The compound matrix $M_{A_\Lambda B_\Gamma}$ is full rank for all matrices $\Lambda \in J[0,1]^{n\times n},\ \Gamma \in J[0,1]^{n\times m}$, [**?**] and the interval compound matrix $\mathcal{M}_{\mathcal{AB}}$ is full rank. If the interval compound matrix $\mathcal{M}_{\mathcal{AB}}$ is full rank, then the system (**??**) is obviously controllable. □

Definition 2.4. The system (**??**) has the observability property on a discrete time-interval $\{0,1,\ldots,k\}$, whenever any pair of input-output sequences (u_i, v_i) , $i = 0,1,\ldots,k$ uniquely specify an initial state x_0.

Definition 2.5. The system (**??**) is observable if for every initial state x_0, there is $k \in \mathbb{N}$ such that the system has observability property on discrete time-interval $\{0,1,\ldots,k\}$.

Definition 2.6. The linear system with interval coefficients by state-space equations in (**??**) is observable if and only if the linear system by state-space equations in (**??**) is observable, for all matrices $\Lambda \in J[0,1]^{n\times n},\ \Gamma \in J[0,1]^{n\times m},\ \Theta \in J[0,1]^{p\times n},\ \Omega \in J[0,1]^{p\times m}$.

Proposition 2.7. *The system (**??**) is observable if and only if the interval compound matrix*

$$N_{\mathcal{CA}} = \begin{bmatrix} \mathcal{C} \\ \mathcal{CA} \\ \mathcal{CA}^2 \\ \vdots \\ \mathcal{CA}^{n-1} \end{bmatrix}$$

is full rank.

Proof. The proof is similar to **??**. □

Example 2.8. Consider the linear system with interval coefficients as :

$$x_{k+1} = \begin{bmatrix} 0 & [2,5] \\ [4,6] & [1,3] \end{bmatrix} x_k + \begin{bmatrix} 7 \\ [-1,3] \end{bmatrix} u_k \qquad\qquad y_k = \begin{bmatrix} 0 & [3,8] \end{bmatrix} x_k \tag{2.3}$$

By using of (**??**), the interval matrices are expressed as follows:
$A_\Lambda = \begin{bmatrix} 0 & 2+3\lambda_{11} \\ 4+2\lambda_{21} & 1+2\lambda_{22} \end{bmatrix}$ such that $A_\Lambda \in \mathcal{A}$ for all matrices $\Lambda \in J[0,1]^{2\times 2}$. $B_\Gamma = \begin{bmatrix} 7 \\ -1+4\gamma_{21} \end{bmatrix}$ such that $B_\Gamma \in \mathcal{B}$ for all matrices $\Gamma \in J[0,1]^{2\times 1}$. $C_\Theta = \begin{bmatrix} & 3+5\theta_{12} \end{bmatrix}$ such that $C_\Theta \in \mathcal{C}$ for all matrices $\Theta \in J[0,1]^{1\times 2}$. $\mathcal{M}_{\mathcal{AB}} = \begin{bmatrix} 7 & [2,5] \\ [-1,3] & [25,51] \end{bmatrix}$, $\mathcal{N}_{\mathcal{CA}} = \begin{bmatrix} 0 & [3,8] \\ [12,48] & [3,24] \end{bmatrix}$. Because of $det(\mathcal{M}_{\mathcal{AB}}) = [130,372]$ and $det(\mathcal{N}_{\mathcal{CA}}) = [-384,-36]$, Then the interval matrices $\mathcal{M}_{\mathcal{AB}}$ and $\mathcal{N}_{\mathcal{CA}}$ are full rank and the system is controllable and observable.

$(\mathcal{M}_{\mathcal{AB}})_{mid} = \begin{bmatrix} 7 & 5 \\ 1 & 38 \end{bmatrix}$, $(\mathcal{M}_{\mathcal{AB}})_{rad} = \begin{bmatrix} 0 & 10 \\ 2 & 13 \end{bmatrix}$. $\rho(\mid (\mathcal{M}_{\mathcal{AB}})^{+}_{mid} \mid (\mathcal{M}_{\mathcal{AB}})_{rad}) = 0.5617 < 1$, the interval matrix $\mathcal{M}_{\mathcal{AB}}$ is full rank by spectral radius criterion. $\sigma_{min}(\mathcal{M}_{\mathcal{AB}})_{mid} = 46.2544$ and $\sigma_{max}(\mathcal{M}_{\mathcal{AB}})_{rad} = 271.5268$, the interval matrix $\mathcal{M}_{\mathcal{AB}}$ is not full rank by singular value criterion. $\parallel (\mathcal{M}_{\mathcal{AB}})^{+}_{mid} \parallel^{-1} = 6.0698$, $\parallel (\mathcal{M}_{\mathcal{AB}})_{rad} \parallel = 15$, the interval matrix $\mathcal{M}_{\mathcal{AB}}$ is not full rank by norm criterion. $(\mathcal{N}_{\mathcal{CA}})_{mid} = \begin{bmatrix} 0 & 5.5 \\ 30 & 13.5 \end{bmatrix}$, $(\mathcal{N}_{\mathcal{CA}})_{rad} = \begin{bmatrix} 0 & 2.5 \\ 18 & 10.5 \end{bmatrix}$. $\rho(|((\mathcal{N}_{\mathcal{CA}})_{mid})^{+}|(\mathcal{N}_{\mathcal{CA}})_{rad}) = 0.6 < 1$, then the interval matrix $\mathcal{N}_{\mathcal{CA}}$ is full rank by spectral radius criterion. $\sigma_{min}((\mathcal{N}_{\mathcal{CA}})_{mid}) = 25.0353$ and $\sigma_{max}((\mathcal{N}_{\mathcal{CA}})_{rad}) = 435.8539$, the interval matrix $\mathcal{N}_{\mathcal{CA}}$ is not full rank by singular value criterion. $\parallel (\mathcal{N}_{\mathcal{CA}})_{mid})^{+} \parallel^{-1} = 5.5$, $\parallel (\mathcal{N}_{\mathcal{CA}})_{rad} \parallel = 28.5$, the interval matrix $\mathcal{N}_{\mathcal{CA}}$ is not full rank by norm criterion.

3. CONCLUSION

IOn this paper, the controllability and observability of discrete time linear system with interval coefficients are investigated. In this theory, no changes apply on the system and both controllability and observability of the system are directly examined. Some criteria are presented that they expressed the being full rank of interval matrix by those criteria and to apply them to $\mathcal{M}_{\mathcal{AB}}$ *,* $\mathcal{N}_{\mathcal{CA}}$*, they also examine the controllability and observability of the system by that criterion. The conditions of controllability and observability of the system by means of the criteria are expaned in the future works.*

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