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APPLICATION OF CONTROL AND OPTIMAL TREATMENT FOR PREDATOR- PREY MODEL

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Abstract. In this paper, a four dimensional predator- prey model with infection in both prey and predator populations is studied. This model consists of susceptible prey, infected prey, susceptible predator, and infected predator. A control model and an optimal treatment are studied. Infact, we make the control model for the above model and analyze the application of control for related system. By this mean, we add the control functions u_1 and u_2 where u_1 is denoted as control factor for prey and infected prey species. u_2 is denoted as control factor for predator and infected predator species. The role of control functions is to treat the infected prey and predator species. Moreover, we briefly describe the optimal control approach. Finally, we are going to control the population of infected prey and predators and establish the optimal criterion that consists of minimizing the total number of infected species along with the treatment costs.

1. Introduction

Infectious diseases can be as a major factor in regulating human population size. For example, Black Death in Europe in the 14^{th} century killed up to one-fourth of the people [3]. Reader may see more appliaction of diseases and control models for some biological models in [1], [2], [5] and [6]. Let $s(t)$, $i(t)$, $x(t)$ and $y(t)$ denote the densities of the susceptible prey, infected prey, susceptible predator and infected predator respectively. A

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four dimensional predator- prey model with infection in both species may be written:

$$\begin{aligned}
\frac{ds}{dt} &= s(1-s-i) - asi - bsx,\\
\frac{di}{dt} &= asi - di - fxi - miy,\\
\frac{dx}{dt} &= csx + gix - exy - \delta x,\\
\frac{dy}{dt} &= exy - \alpha y + niy. \qquad (1.1)
\end{aligned}$$

This model consists of two populations: the prey population and the predator population. Both of the presented populations have two sub classes: susceptible and infected [4]. It is assumed that in this research all parameters are non-negative.

2. Control Model Analysis

We make the control model for the above system (1.1) and analyze it. By adding u_1(control factor for prey and infected prey species) and u_2 (control factor for the predator and infected predator species) we may write the following control system :

$$\begin{aligned}
\frac{ds}{dt} &= s(1-s-i) - a(1-u_1)si - bsx,\\
\frac{di}{dt} &= a(1-u_1)si - di - fxi - miy,\\
\frac{dx}{dt} &= csx + gix - e(1-u_2)xy - \delta x,\\
\frac{dy}{dt} &= e(1-u_2)xy - \alpha y + niy, \qquad (2.1)
\end{aligned}$$

where $0 \leqslant\ u_i \leqslant 1, i = 1, 2$. When $u_i = 0$, no treatment occurs and whenever $u_i = 1$, the model shows the full treatment.

Theorem 2.1. *Nontrivial equilibrium point E^* is locally asymptotically stable for model (2.1) provided $D'_1 > 0, D'_3 > 0, D'_4 > 0$ and $D'_1 D'_2 D'_3 > {D'_3}^2 + {D'_1}^2 D'_4$. where $D'_1, D'_2, D'_3$and D'_1 are assumed as coefficiets of corresponding characteristic equation.*

Proof. At the interior equlibrium E^*, the Jacobian matrix $J(E^*)$ can be obtained as follows:

$$J(E^*) = \begin{pmatrix} -s^* & -(1+a(1-u_1))s^* & -bs^* & 0 \\ a(1-u_1)i^* & 0 & -fx^* & -mi^* \\ cx^* & gx^* & 0 & -e(1-u_2)x^* \\ 0 & ny^* & e(1-u_2)y^* & 0 \end{pmatrix}$$

The corresponding characteristic equation is $\lambda^4 + D'_1\lambda^3 + D'_2\lambda^2 + D'_3\lambda + D'_4 = 0$ where $D'_1 = s^*, D'_2 = e^2(1-u_2)^2x^*y^* + fgx^{*^2} + mni^*y^* + a(1-u_1)[a(1-u_1)+1]s^*i^* + bcs^*x^*$, $D'_3 = e^2(1-u_2)^2s^*x^*y^* + fgs^*x^{*2} + mns^*i^*y^* - e(1-u_2)nfx^{*2}y^* + me(1-u_2)$ $gi^*x^*y^* + a(1-u_1)bgs^*i^*x^* - cf(a(1-u_1)+1)s^*x^*$, and $D'_4 = -e(1-u_2)nfs^*x^{*2}y^* +$

$[e(1-u_2)mg+a(1-u_1)(a(1-u_1)+1)e(1-u_2)-a(1-u_1)be(1-u_2)n+bcmn-cme(1-u_2)[a(1-u_1)+1)]s^*i^*x^*y^*$.

By Routh - Hurwitz criterion, all the eigenvalues of $J(E^*)$ have negative real parts if

$$\begin{cases} D_i' > 0, i = 1,3,4, \\ D_1'(D_2'D_3' - D_1'D_4') - D_3'^2 > 0. \end{cases}$$

Therefore, if $D_1' > 0, D_3' > 0, D_4' > 0$ and $D_1'D_2'D_3' > {D_3'}^2 + {D_1'}^2 D_4'$, E^* is locally asymptotically stable. □

3. Optimal Control

We now study the optimal control approach. Infact, we are going to control the population of infected prey and predators. We establish the optimal criterion that consists of minimizing the total number of infected species along with the treatment costs. Following control function is necessary that minimizes the objective functional:

$$Min \int_0^T [\frac{1}{2}(W_1{u_1}^2 + W_2{u_2}^2) + i(t) + y(t)]dt$$

We derive the associated Hamiltonian for the optimal control problem:

$$\begin{gathered} H = \tfrac{1}{2}(W_1{u_1}^2 + W_2{u_2}^2) + i(t) + y(t) + p_s(t)(s(1-s-i) - a(1-u_1)si - bsx) \\ +p_i(t)(a(1-u_1)si - di - fxi - miy + p_x(t)(csx + gix - e(1-u_2)xy - \delta x) \\ +p_y(t)(e(1-u_2)xy - \alpha y + niy). \end{gathered}$$

where the functions $p_s(t), p_i(t), p_x(t), p_y(t)$ are called the co-state variables. These variables must satisfy the following set of differential equations

$$\frac{dp_s(t)}{dt} = -\frac{\partial H}{\partial s}, \frac{p_i(t)}{dt} = -\frac{\partial H}{\partial i}, \frac{dp_x(t)}{dt} = -\frac{\partial H}{\partial x}, \frac{dp_y(t)}{dt} = -\frac{\partial H}{\partial y}$$

and so

$$\begin{aligned} \frac{dp_s(t)}{dt} &= -(p_s(t) - 2sp_s(t) - ip_s(t) - aisp_s(t) + aisu_1p_s(t) - bxp_s(t)), \\ \frac{dp_i(t)}{dt} &= -(1 - sp_s(t) - asp_s(t) + asu_1p_s(t) + asp_i(t) - asu_1p_i(t) - dp_i(t) \\ &\quad - fxp_i(t) - mip_i(t) + gxp_x(t) + nyp_y(t)), \\ \frac{dp_x(t)}{dt} &= -(-bsp_s(t) - fip_i(t) + csp_x(t) + gip_x(t) - eyp_x(t) + eyu_2p_x(t) \\ &\quad - \delta p_x(t) + eyp_y(t) - u_2eyp_y(t)), \\ \frac{dp_y(t)}{dt} &= -(1 - mip_i(t) - exp_x(t) + exu_2p_x(t) + exp_y(t) - exu_2p_y(t) - \alpha \\ &\quad p_y(t) + nip_y(t)). \end{aligned} \tag{3.1}$$

We have the boundary conditions for system (3.1) as follows,

$$s(0) = s_\circ, i(0) = i_\circ, x(0) = x_\circ, y(0) = y_\circ, p_s(T) = p_i(T) = p_x(T) = p_y(T) = 0.$$

Differentiating H with respect to u_1 and u_2 yields:

$$\frac{\partial H}{\partial u_1} = W_1 u_1 + asi(p_s(t) - p_i(t)) = W_1 u_1 + (asi)p_s(t) - (asi)p_i(t)) = 0$$
$$\frac{\partial H}{\partial u_2} = W_2 u_2 + exi(p_x(t) - p_y(t)) = W_2 u_2 + (exi)p_x(t) - (exi)p_y(t)) = 0$$

Thus, the control factors u_1 and u_2 can be obtained as follows:

$$u_1(t) = \frac{asi}{W_1}(-p_s(t) + p_i(t)), u_2(t) = \frac{exi}{W_2}(-p_x(t) + p_y(t)).$$

As regards $0 \leqslant u_1, u_2 \leqslant 1$, we have the optimal control laws as:

$$u_1{}^*(t) = \min\{1, \max\{0, u_1\}\} = \min\{1, \max\{0, \frac{asi}{W_1}(p_i(t) - p_s(t))\}\},$$
$$u_2{}^*(t) = \min\{1, \max\{0, u_2\}\} = \min\{1, \max\{0, \frac{exi}{W_2}(p_y(t) - p_x(t))\}\}.$$

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