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SOLVING NONLINEAR FRACTIONAL LANGEVIN EQUATION BY CAS WAVELETS

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Abstract. This paper presents a computational method for solving nonlinear fractional Langevin equation(NFLE). First we convert the NFLE into a system of nonlinear algebraic equations. Then by solving this system, the approximate solution of the main problem is obtained. Finally, we solve an example by the proposed method and then approximate and exact solutions are compared.

1. Introduction

In 1908, the Langevin equation was proposed by French phisicist Paul Langevin to give an elaborate description of Brownian motion [2]. The existence and uniqueness of solutions of the fractional Langevin equation were verified by Baghani [1] and Yu et al.[5]. In this paper, we use CAS wavelets to solve the initial value problem of nonlinear fractional Langevin equation of the form

$$D^{\beta}(D^{\alpha}+\gamma)x(t)=f(t,x(t)), \qquad 0<t\leq 1, \tag{1.1}$$

$$x(0)=\mu, \qquad x^{(\alpha)}(0)=\nu,$$

where $f(t,x(t))=\sum_{j=0}^{n} a_j x^j(t)+g(t)$, $a_j\in\mathbb{R}$ for $0\leq j\leq n$, $\gamma\in\mathbb{R}$, $0<\alpha<1$, $0<\beta<1$, D^{α} and D^{β} are the Caputo derivatives and $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuously differentiable function.

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The Caputo fractional derivative of order α, $k-1<\alpha\leq k, k\in\mathbb{N}$, of the function $f(t)$ is defined as

$$D^{\alpha}f(t)=I^{k-\alpha}D^{k}f(t)=\frac{1}{\Gamma(k-\alpha)}\int_{a}^{t}\frac{f^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}}d\tau$$

where $I^{\beta}(.)$ is the Riemann-Liouville fractional integral of order β.

2. CAS wavelets and operational matrix of the fractional integration

CAS wavelets are defined on the interval $[0,1)$ as

$$\psi_{ij}(t)=\begin{cases}2^{\frac{k}{2}}CAS_j(2^k t-i+1), & \frac{i-1}{2^k}\leqslant t<\frac{i}{2^k},\\ 0, & \text{otherwise,}\end{cases}$$

where $i=1,\dots,2^k$, $j\in\mathbb{Z}$, $k\in\mathbb{N}$, and $CAS_j(t)=cos(2j\pi t)+sin(2j\pi t)$.

A function $f(t)\in L^2[0,1)$ can be approximated as

$$f(t)\approx\sum_{i=1}^{2^k}\sum_{j=-M}^{M}c_{ij}\psi_{ij}(t)=C^T\Psi(t)=\widehat{f}(t), \tag{2.1}$$

where C and $\Psi(t)$ are $2^k(2M+1)\times 1$ matrices.

Assume that $m=2^k(2M+1)$ and $t_i=\frac{i-1}{m}$, $i=1,2,\dots m$. Define CAS wavelets matrix as

$$\Phi_{m\times m}=\begin{bmatrix}\Psi(t_1), & \Psi(t_2), & \cdots, & \Psi(t_m)\end{bmatrix}.$$

It is obvious that $C^T=[\hat{f}(t_1),\hat{f}(t_2),\cdots,\hat{f}(t_m)]\Phi_{m\times m}^{-1}$.

It is known that, any square integrable function $f(t)$ on $[0,1)$ can be approximated in terms of block puls functions (BPFs) [3] as

$$f(t)\approx\sum_{i=0}^{m-1}f_i b_i(t)=F^T B_m(t),$$

where $F=[f_0,\dots,f_{m-1}]^T$ and $B_m(t)=[b_0(t),\dots,b_{m-1}(t)]^T$.

CAS wavelets can be expanded in terms of BPFs as follows

$$\Psi(t)=\Phi_{m\times m}B_m(t). \tag{2.2}$$

Note that, we have the following theorem from [3]:

Theorem 2.1. *A function $u(t)\in L^2[0,1]$ with bounded second derivative, $\left|u^{(2)}(t)\right|\leqslant\gamma$, can be expanded as an infinite sum of the CAS wavelets and the series converges uniformly to $u(t)$, that is $u(t)=\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}c_{nm}\psi_{nm}(t)$. Furthermore, we have*

$$\left|u(t)-\sum_{n=1}^{2^k}\sum_{m=-M}^{M}c_{nm}\psi_{nm}(t)\right|\leqslant\frac{\gamma}{\pi^2}\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{5/2}m^2},\quad t\in[0,1].$$

Let $(I^{\alpha}\Psi)(t) \approx P^{\alpha}_{m\times m}\Psi(t)$, where I^{α} is the Riemann-Liouville fractional integral operator of order α. The matrix $P^{\alpha}_{m\times m}$ is called the CAS wavelets operational matrix of the fractional integration and is obtained as

$$P^{\alpha}_{m\times m} = \Phi_{m\times m} F^{\alpha} \Phi^{-1}_{m\times m},$$

where F^{α} is the BPFs operational matrix of fractional integration[4] and $I^{\alpha}B_m(t) \approx F^{\alpha}B_m(t)$.

3. Method of numerical solution

Consider the initial value problem (1.1) of Langevin equation. From [5], we can see that $x(t)$ is a solution of Eq. (1.1) if and only if $x(t)$ is a solution of the integral equation:

$$x(t) = I^{\alpha+\beta} f(t, x(t)) - \gamma I^{\alpha} x(t) + h(t) \tag{3.1}$$

where

$$h(t) = \mu + \frac{\nu + \gamma\mu}{\Gamma(\alpha+1)} t^{\alpha}.$$

Assume that $x(t) \simeq X^T\Psi(t) = (X^T\Phi)B(t) = (X^T\Phi)_1 B(t)$. Then for $j = 2, \ldots, n$, we have

$$\begin{aligned} x^j(t) &= \underbrace{X^T\Psi(t) \times \ldots \times X^T\Psi(t)}_{j-times} = X^T\Phi B(t) \times \ldots \times X^T\Phi B(t) \\ &= (X^T\Phi \otimes \ldots \otimes X^T\Phi)B(t) = (X^T\Phi)_j B(t). \end{aligned}$$

Thus $x^j(t) = (X^T\Phi)_j B(t)$ for $j = 1, 2, \ldots, n$. Note that, if $A = (a_i)$ and $B = (b_i)$ then $A \otimes B = (a_i b_i)$. Let $g(t) \simeq G^T\Psi(t)$. We obtain

$$\begin{aligned} I^{\alpha+\beta} f(t, x(t)) &= I^{\alpha+\beta}(\sum_{j=0}^{n} a_j x^j(t) + g(t)) = \sum_{j=0}^{n} a_j I^{\alpha+\beta}(x^j(t)) + I^{\alpha+\beta}(g(t)) \\ &= \sum_{j=0}^{n} a_j I^{\alpha+\beta}((X^T\Phi)_j B(t)) + I^{\alpha+\beta}(G^T\Psi(t)) = \sum_{j=0}^{n} a_j (X^T\Phi)_j I^{\alpha+\beta}B(t) + G^T I^{\alpha+\beta}\Psi(t) \\ &= \sum_{j=0}^{n} a_j (X^T\Phi)_j F^{\alpha+\beta}B(t) + G^T P^{\alpha+\beta}\Psi(t) = (\sum_{j=0}^{n} a_j (X^T\Phi)_j F^{\alpha+\beta} + G^T P^{\alpha+\beta}\Phi)B(t). \end{aligned}$$

Therefore we have

$$I^{\alpha+\beta} f(t, x(t)) = (\sum_{j=0}^{n} a_j (X^T\Phi)_j F^{\alpha+\beta} + G^T P^{\alpha+\beta}\Phi)B(t) = A^T B(t). \tag{3.2}$$

Now assume that $h(t) = H^T\Psi(t) = H^T\Phi B(t)$. Since $I^{\alpha}x(t) = X^T P^{\alpha}\Phi B(t)$, then by substituting this, $x(t) = X^T\Phi B(t)$, $h(t) = H^T\Phi B(t)$ and Eq. (3.2) in Eq. (3.1) we will have

$$X^T\Phi B(t) = A^T B(t) - \gamma X^T P^{\alpha}\Phi B(t) + H^T\Phi B(t).$$

Eliminating $B(t)$ of both sides of the last equation yields the system

$$X^T\Phi = A^T - \gamma X^T P^\alpha \Phi + H^T \Phi$$

of nonlinear algebraic equations. By solving this system, the approximate solution of equation (3.1) is obtained.

4. Numerical example

Consider the nonlinear fractional Langevin equation

$$D^{5/8}(D^{1/5}+1)x(t) = x^2(t) + g(t), \tag{4.1}$$

with initial conditions $x(0) = 0, x^\alpha(0) = 0$. Also $g(t) = -t^6 + \frac{6}{\Gamma(\frac{127}{40})}t^{\frac{87}{40}} + \frac{6}{\Gamma(\frac{27}{8})}t^{\frac{19}{8}}$. The exact solution of this equation is $X(t) = t^3$. We have solved the Eq. (4.1) using CAS wavelets method. Table 1 shows the absolute error $E_{k,M}(t) =\mid x(t) - x_{k,M}(t) \mid$ between exact solution $x(t)$ and approximate solution $x_{k,M}(t)$ in different values of t.

Table 1. Absolute error for $M = 2$ and $k = 3, 4, 5$

t	$M=2, k=3$	$M=2, k=4$	$M=2, k=5$
0.1	$2.5178e-05$	$6.1366e-06$	$1.4986e-06$
0.2	$4.7952e-05$	$1.1674e-05$	$2.8485e-06$
0.3	$6.9872e-05$	$1.7002e-05$	$4.1468e-06$
0.4	$9.1602e-05$	$2.2282e-05$	$5.4333e-06$
0.5	$1.1387e-04$	$2.7692e-05$	$6.7516e-06$
0.6	$1.3768e-04$	$3.3475e-05$	$8.1606e-06$
0.7	$1.6446e-04$	$3.9982e-05$	$9.7460e-06$
0.8	$1.9640e-04$	$4.7740e-05$	$1.1636e-05$
0.9	$2.3690e-04$	$5.7576e-05$	$1.4033e-05$
CPU time(s)	4.549452	13.817545	75.781281

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