National Seminar on Control and Optimization
Nov 13-14, 2019
سومین سمینار ملی
کنترل و بهینه‌سازی
۲۲-۲۳ آبان ۱۳۹۸

SCW METHOD FOR SOLVING THE FRACTIONAL LINEAR LANGEVIN EQUATION

ESMAIL BRAGAMADI [1], MOHAMMAD TAGHI KHODADAD[2*]

[1,2] *Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.*
esmailbargamadi@gmail.com; khodadad44@gmail.com

Abstract. In this paper, we use the second Chebyshev wavelets (SCWs) for solving the linear fractional Langevin equation. By using the second Chebyshev wavelets operational matrix, the fractional Langevin equation (FLE) is transformed into a system of algebraic equations. Then by solving this algebraic system, the approximate solution of the main problem is obtained. Finally, we solve an example by the SCWs method.

1. Introduction

In 1908, the Langevin equation was proposed by French phisicist Paul Langevin to give an elaborate description of Brownian motion[2]. The existence and uniqueness of solutions of the fractional Langevin equation were verified by Tao Yi et al.[5] and Baghani [1]. in this paper, we use the second Chebyshev wavelets to solve the initial value problem of fractional Langevin equation of the from

$$\begin{aligned} &D^{\beta}(D^{\alpha}+\gamma)x(t)=f(t), \qquad 0<t\leq 1, \\ &x(0)=\mu, \quad x^{(\alpha)}(0)=\nu, \end{aligned} \tag{1.1}$$

where $\gamma \in \mathbb{R}$, $0<\alpha<1$, $0<\beta<1$, D^{α} and D^{β} are the Caputo derivatives and $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuously differentiable function.

The Caputo fractional derivative of order α, $k-1<\alpha\leq k, k\in\mathbb{N}$, of the function $f(t)$ is defined as

$$D^{\alpha}f(t)=I^{k-\alpha}D^{k}f(t)=\frac{1}{\Gamma(k-\alpha)}\int_{a}^{t}\frac{f^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}}d\tau$$

2010 *Mathematics Subject Classification.* 34A08, 26A33.

Key words and phrases. Fractional Langevin equation, Chebyshev wavelets, Operational matrix, Riemann-Liouville fractional integration, Caputo derivative.

* Speaker.

where $I^\beta(.)$ is the Riemann-Liouville fractional integral of order β.

2. The SCWs and operational matrix of the fractional integration

The second Chebyshev wavelets are defined on the interval $[0, 1)$ as:

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}U_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 1, 2, \dots, 2^{k-1}$, $m = 0, 1, \dots, M - 1$, k and M are positive integers and coefficient $\sqrt{\frac{2}{\pi}}$ is used for orthonormalitly. The function $U_m(t)$ is the second Chebyshev polynomial of degree m. Note that, These polynomials are defined on the interval $[-1, 1]$ by the recurrence

$$U_0(t) = 1, \qquad U_1(t) = 2t, \qquad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t),$$

where $m = 1, 2, \dots$.

A function $f \in L^2([0, 1])$ can be approximate in terms of the SCWs as

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t) = \hat{f}(t) \tag{2.1}$$

where

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \dots, \psi_{1(M-1)}(t), \psi_{20}(t), \dots, \psi_{2(M-1)}(t), \dots, \psi_{2^{k-1}0}(t), \dots, \psi_{2^{k-1}(M-1)}(t)]^T,$$

$$C = [c_{10}, c_{11}, \dots, c_{1(M-1)}, c_{20}, \dots, c_{2(M-1)}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}(M-1)}]^T.$$

We define the SCWs matrix $\Phi_{m'\times m'}$ as

$$\Phi_{m'\times m'} = [\Psi(\frac{1}{2m'}), \Psi(\frac{3}{2m'}), \dots, \Psi(\frac{2m'-1}{2m'})],$$

where $m' = 2^{k-1}M$.

Theorem 2.1. *Suppose that $f : [0, 1] \longrightarrow \mathbb{R}$ be a real valued function and $f \in C^m[0, 1]$. Then we have*

$$\| \hat{f}(t) - f(t) \| \leq \frac{2}{2^{m'(k-1)}4^{m'}m!} \sup_{t\in[0,1]} | f^{(m)}(t) |$$

where $\hat{f}(t)$ is given by Eq. (2.1).

Proof. see [3] □

Now, we define a m'-set of block-pulse functions (BPFs) on the interval $[0, 1)$ as

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m'} \leq t < \frac{i}{m'} \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 1, ..., m'$.

Also, a function $f \in L^2([0,1])$ can be approximate in terms of BPFs as the from

$$f(t) \approx \sum_{i=1}^{m'} f_i b_{m'}(t) = F^T B_{m'}(t)$$

where $F = [f_1, f_2, ..., f_{m'}]^T$ and $B_{m'}(t) = [b_1(t), b_2(t), ..., b_{m'}(t)]^T$.

Chebyshev wavelets can be expanded in terms of BPFs as

$$\Psi(t) = \Phi_{m' \times m'} B_{m'}(t).$$

Let

$$I^\alpha \Psi(t) \approx P^\alpha{}_{m' \times m'} \Psi(t), \tag{2.2}$$

where I^α is the Riemann-Liouville fractional integral operator of order α. The matrix $P^\alpha_{m' \times m'}$ is called the Chebyshev wavelets operational matrix of fractional integration and is obtianed as follows

$$P^\alpha{}_{m' \times m'} = \Phi_{m' \times m'} F^\alpha \Phi^{-1}_{m' \times m'},$$

where F^α is the BPFs operational matrix of fractional integration [4] and $I^\alpha B_{m'}(t) = F^\alpha B_{m'}(t)$.

3. Solving fractional linear Langevin equation

Consider the initial value problem (1.1). From [5], we can see that $x(t)$ is a solution of Eq. (1.1) if and only if $x(t)$ is a solution of the integral equation:

$$x(t) = I^{\alpha+\beta} f(t) - \gamma I^\alpha x(t) + g(t) \tag{3.1}$$

where

$$g(t) = \frac{\nu + \gamma\mu}{\Gamma(\alpha+1)} t^\alpha + \mu.$$

Now let

$$x(t) \simeq X^T \Psi(t), \quad f(t) \simeq F^T \Psi(t), \quad g(t) \simeq G^T \Psi(t). \tag{3.2}$$

Then by Eq.(2.2) we have

$$I^{\alpha+\beta} f(t) \simeq F^T P^{\alpha+\beta} \Psi(t), \qquad I^\alpha x(t) \simeq X^T P^\alpha \Psi(t). \tag{3.3}$$

By substituting the Eqs. (3.2) and (3.3) in Eq. (3.1) we obtain

$$X^T \Psi(t) = F^T P^{\alpha+\beta} \Psi(t) - \gamma X^T P^\alpha \Psi(t) + G^T \Psi(t). \tag{3.4}$$

The Eq. (3.4) yields the following system

$$X^T = F^T P^{\alpha+\beta} - \gamma X^T P^\alpha + G^T$$

of algebraic equations. By solving this system, the approximate solution of equation (3.1) is obtained.

4. Numerical example

In this section, we solve an example by proposed method.

Example 4.1. Consider the initial value problem

$$D^{1/2}(D^{1/3}+1)x(t) = f(t), \quad x(0) = 0.5, \quad x^{(\frac{1}{3})}(0) = 0, \tag{4.1}$$

where

$$f(t) = -\frac{\Gamma(2)}{\Gamma(\frac{5}{2})}t^{(\frac{3}{2})} - \frac{10\Gamma(2)\Gamma(5/3)}{3\Gamma(\frac{8}{3})\Gamma(\frac{13}{6})}t^{(\frac{7}{6})}$$

The exact solution of Eq. (4.1) is $x(t) = \frac{1}{2} - t^2$. we have solved the Eq. (4.1) using proposed method. Table 1 shows the absolute error and approximate solutions in different values of t.

Table 1. Approximate solutions and absolute error for $M = 5$ and $k = 5, 7$ in example 4.1

	approximate solutions			absolute error	
t	$M=5, k=5$	$M=5, k=7$	exact solution	$M=5, k=5$	$M=5, k=7$
0.1	$4.899622e-01$	$4.899952e-01$	$4.90e-01$	$3.770139e-05$	$4.777029e-06$
0.2	$4.599650e-01$	$4.599964e-01$	$4.60e-01$	$3.499037e-05$	$3.540749e-06$
0.3	$4.099681e-01$	$4.099970e-01$	$4.10e-01$	$3.180331e-05$	$2.937636e-06$
0.4	$3.399704e-01$	$3.399974e-01$	$3.40e-01$	$2.951829e-05$	$2.577967e-06$
0.5	$2.499721e-01$	$2.499976e-01$	$2.50e-01$	$2.782651e-05$	$2.335707e-06$
0.6	$1.399734e-01$	$1.399978e-01$	$1.40e-01$	$2.651622e-05$	$2.159511e-06$
0.7	$9.974536e-03$	$9.997975e-03$	$1.00e-02$	$2.546331e-05$	$2.024465e-06$
0.8	$-1.400245e-01$	$-1.400019e-01$	$-1.40e-01$	$2.459249e-05$	$1.916949e-06$
0.9	$-3.100238e-01$	$-3.100018e-01$	$-3.10e-01$	$2.385570e-05$	$1.828849e-06$
CPU times(s)	0.229319	1.051019			

References

1. O. Baghani, *On fractional Langevin equation involving two fractional orders*, Commun Nonlinear Sci Numer Simulat 42 (2017) 675-681.
2. P. Langevin, *On the theory of Brownian motion*, CR Acad Sci, 1908.
3. Y. Wang,Q. Fan, *The second kind Chebyshev wavelets method for solving fractional differential equations*, Appl. Math. Comput, 218 (2012) 8592-8601.
4. Y. Wang, L. Zhu, *SCW method for solving the fractional integro-differential equations with a weakly singular kernel*, Appl. Math. Comput, 275 (2016) 72-80.
5. T. Yu, K. Deng, M. Luo, *Existence and uniqueness of solutions of initial value problems for nonlinear langevin equation involving two fractional orders*, Commun Nonlinear Sci Numer Simulat 19 (2014) 1661-1668.