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THE MOST PROBABLE ALLOCATION SOLUTION FOR THE P-MEDIAN PROBLEM

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Abstract. In this paper, a minimum information approach is applied to the capacitated p-median problem to estimate the most likely allocation solution. Indeed, the most probable solution is achieved through minimizing a log-based objective function while the total transportation cost should be less than or equal to a pre-determined budget. The problem is solved by using the Lagrangian dual method and a numerical example is provided to verify the added value of the proposed model.

1. Introduction

The p-median problem is one of the main issues in the field of locating facilities with a mini-sum objective function where its purpose is to locate p facilities and allocate demand nodes so that the total demand-weighted distance would be minimized. Various approaches to solve the p-median problem have been provided by [4]. When there is a lack of information in the network, the minimum information (MI) approach can be applied, as an extension of the well-known entropy model [5] which provides an extended measure of the likelihood of a certain macro state on the existence of some appropriate micro state space.

Abareshi and Zaferanieh [1] proposed a new bi-level p-median problem in which the total cost of locating facilities as well as serving demands was minimized through the upper level while the MI approach was applied in the lower level to determine the most unbiased allocation solution. In this paper, we attempt to locate p capacitated facilities with a pre-determined budget where the demands are not only allocated based on the distances, but some other attributes such as local and geographical features represented by prior probabilities of serving the demand of nodes by different facilities, taken into account by multiple attributes decision making procedures are also effective.

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Table 1. The notation used in the model

Notation	Definition
N	$=\{v_1,...,v_n\}$ The set of nodes
E	The set of links
I	The index set of client nodes
J	The index set of candidate points for establishing facilities
y_j	The binary variable representing whether the node $v_j, j\in J$ is used to locate a facility or not
c_j	The capacity of node $v_j, j\in J$
w_i	The total demand of the client node $v_i,\ i\in I$
p_{ij}	The probability of serving the demand of node $v_i, i\in I$ by facility $v_j, j\in J$
x_{ij}	The decision variable representing the amount of demand of node $v_i, i\in I$ provided by facility $v_j, j\in J$
d_{ij}	Per unit cost to serve the demand of node $v_i, i\in I$ by facility $v_j, j\in J$
B	The total available budget

2. The capacitated p-median problem applying the minimum information approach

Consider a network $G=(N,E)$ where the other frequently used notations are listed in Table (1). Following the approach presented in [1] to formulate the log-based objective function resulted by the MI, the minimum information capacitated p-median ($MICpM$) problem is introduced as follows:

$$\text{(MICpM):}\ \min_{x,y}\sum_{j\in J}\sum_{i\in I}x_{ij}(\ln x_{ij}-1-\ln p_{ij}) \tag{2.1}$$

$$\sum_{j\in J}y_j=p \tag{2.2}$$

$$\sum_{j\in J}x_{ij}=w_i,\ \forall i\in I \tag{2.3}$$

$$\sum_{i\in I}x_{ij}\le y_jc_j,\ \forall j\in J \tag{2.4}$$

$$\sum_{j\in J}\sum_{i\in I}d_{ij}x_{ij}\le B \tag{2.5}$$

$$y_j\in\{0,1\},\ x_{ij}\ge 0,\ \forall i\in I, j\in J,$$

Constraint (2.2) assures that the number of established facilities is equal to p and constraints (2.3) practically guarantee that the demands of all vertices i are supplied. Constraints (2.4) make all open facilities serve the demands less than or equal to their capacities. In addition, Constraint (2.5) states that the total spent cost does not exceed the available budget B.

2.1. **Lagrangian dual solution approach.** Relaxing constraints (2.4) by introducing the dual variables $\lambda_j\ge 0$, the location aspect of the problem can be separated from the rest by using the following Lagrangian dual problem, see [2]:

$$\max_{\lambda} LR(\lambda) = \min_{x,y} L(x,y) = \sum_{j\in J}\sum_{i\in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij}) + \sum_{j\in J}\lambda_j(\sum_{i\in I} x_{ij} - y_j c_j)$$
$$= \sum_{j\in J}\sum_{i\in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij} + \lambda_j) - \sum_{j\in J}\lambda_j c_j y_j \qquad (2.6)$$

subject to binary and non-negativity constraints with (2.2),(2.3) and (2.5). Given multipliers $\lambda_j \geq 0$, Problem $L(x,y)$ can be decomposed into two following sub-problems:

$$SP_1 : \max_{y} L_1(y) = \sum_{j\in J}\lambda_j c_j y_j \qquad (2.7)$$
$$s.t. \ \sum_{j\in J} y_j = p, \ y_j \in \{0,1\} \ \forall j \in J$$

Clearly, having the variables λ_j, the optimal solution of SP_1 would be estimated by selecting the p largest values $\lambda_j c_j$ and setting the corresponding variables $y_j = 1$ and the others equal to zero. The second sub-problem is introduced as follows:

$$SP_2 : \min_{x} L_2(x) = \sum_{j\in J}\sum_{i\in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij} + \lambda_j) \qquad (2.8)$$
$$\sum_{j\in J} x_{ij} = w_i, \ \forall i \in I \qquad (2.9)$$
$$\sum_{j\in J}\sum_{i\in I} d_{ij} x_{ij} \leq B \qquad (2.10)$$
$$x_{ij} \geq 0, \ \forall i \in I, j \in J.$$

Lemma 2.1. *Let $S = \{j \in J | y_j = 1\}$ be the set of located facilities determined by sub-problem SP_1 and γ, η be the Lagrangian dual multipliers corresponding to constraints (2.9) and (2.10) in the sub-problem SP_2. Then the optimal solution of (2.8) is estimated by $x_{ij} = p_{ij} e^{-(\lambda_j + \gamma_i + \eta d_{ij})}$, $\forall j \in S, i \in I$, see [1].*

Writing the Karush-Kuhn-Tucker (KKT) optimality conditions of model (2.8) and using the linearized approaches introduced in [3], the linear mixed-integer system to be solved is equivalent to the following:

$$x_{ij} = v_j f_{ij}, f_{ij} \leq M y_j, \ f_{ij} \geq z_{ij} - M(1 - y_j), \ f_{ij} \leq z_{ij} \ \forall i \in I, \ j \in J$$
$$B - \sum_{j\in J}\sum_{i\in I} x_{ij} \leq Mt, \ u_i - z_{ij} \leq M(1-t), \ \forall i \in I, \forall j \in J$$
$$z_{ij} \leq u_i, \ k_i \geq 1 \ \forall i \in I, \ j \in J, \sum_{j\in J}\sum_{i\in I} x_{ij} \leq B, \ \sum_{j\in J} x_{ij} = w_i, \ \forall i \in I$$
$$u_i, z_{ij} > 0, f_{ij} \geq 0, \ t \in \{0,1\}, \ \forall i \in I, \ j \in J,$$

in which M is a sufficiently large number and $v_j = e^{-\lambda_j}$ where λ_js could be estimated through iterative methods [2]. To obtain the tightest lower bound that is close to the

TABLE 2. The nodes' capacities and demands along with their weights with respect to criteria $r = 2, ..., 6$

Nodes	1	2	3	4	5	6	7	8	9
c	300	200	350	320	250	250	350	100	200
w	50	60	80	40	90	100	80	60	50
Criteria	2	3	4	5	6	7	8	9	
2	0.0541	0.0270	0.1351	0.1622	0.0811	0.1081	0.0541	0.1351	0.2432
3	0.0244	0.0488	0.1220	0.0976	0.1707	0.1951	0.2195	0.0732	0.0488
4	0.0455	0.1136	0.1591	0.0227	0.0682	0.1591	0.2045	0.1818	0.0455
5	0.2162	0.0270	0.0811	0.1351	0.1081	0.0541	0.2432	0.0270	0.1081
6	0.2286	0.0571	0.1429	0.0286	0.1143	0.0857	0.2571	0.0571	0.0286

TABLE 3. The allocation solution for the $MICpM$ problem

Facility \ Node	1	2	3	4	5	6	7	8	9	$ULOF$
$B = 15000$	$Weight_1 = [0.3462\ 0.1923\ 0.1154\ 0.0385\ 0.1154\ 0.1923]$									
3	15.88	24.89	40.59	7.19	25.27	39.25	11.86	11.85	15.48	2569.23
5	13.39	21.60	22.26	13.38	37.26	36.29	16.81	20.36	14.11	
7	20.72	13.52	17.15	19.43	27.47	24.46	51.33	27.79	20.41	
$B = 10000$	$Weight_1 = [0.3462\ 0.1923\ 0.1154\ 0.0385\ 0.1154\ 0.1923]$									
3	17.73	33.83	76.55	2.48	15.32	48.56	1.29	4.77	15.48	2724.47
5	11.75	23.05	1.00	13.17	64.41	44.90	5.20	23.36	14.11	
7	20.52	3.12	2.45	24.35	10.27	6.54	73.51	31.88	20.41	

optimal value, the problem $LR(\lambda^*) = \max_{\lambda \geq 0} LR(\lambda)$ should be solved. The sub-gradient algorithm is an effective method to solve this Lagrangian dual problem whereby the lower and upper bounds are updated, see [2].

Example 2.2. Consider the small network shown in Figure (1) wherein $I = J = N$ and the costs of links have been given next to them. The amount of d_{ij} for each pair (i, j) is considered as the shortest distance between i and j. In addition, the capacities, c, and demands, w, along with the comparative weights of nodes with respect to the other 5 attributes $r = 2, ..., 6$, except the approximate distances, to compute the probabilities p_{ij}, are inserted in Table (2), see [1].

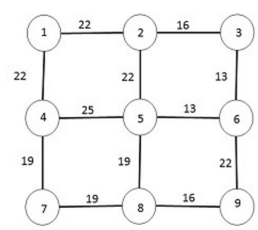

FIGURE 1. The grid network with 9 nodes

To see the effects of available budget as well as emphasizing on different attributes, the solution of the $MICpM$ problem for $p = 3$ is estimated for $B = 15000$ and $B = 10000$ and weight vector $Weight_1$ inserted in Table (3). It can be realized, if capacities are large enough, the shares of selected facilities for the $MICpM$ allocation solution are proportional to their probabilities, see Table (3). The probabilities $p(1, j); j = 1, ..., 9$ for $Weight_1$ are estimated as $p(1, :) = [0.1617,\ 0.1072,\ 0.1090,\ 0.1423,\ 0.0983,\ 0.0895,\ 0.1471,\ 0.0642,\ 0.0808]$.

Among selected medians $3, 5, 7$ in Table (3), median 7 has the most probability to serve the demand of node 1 while nodes 3 and 5 are ranked as the second and third. This can be also inferred from Table (3). However, decreasing the budget B to 10000 results in a different allocation attempting to assign as much demand as possible to the closer facilities, compare the column of $i = 3$ in cases $B = 10000$ and $B = 15000$.

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