National Seminar on Control and Optimization 3th
سومین سمینارملی
کنترل وبهینه سازی

ON DELAY FRACTIONAL OPTIMAL CONTROL PROBLEMS WITH A COMBINATION OF CONFORMABLE AND CAPUTO-FABRIZIO FRACTIONAL DERIVATIVES

FARZANEH KHEYRINATAJ[1] AND ALIREZA NAZEMI[1*]

[1] *Faculty of Mathematical Sciences,*
Shahrood University of Technology, Shahrood, Iran
farzane_kheyrinataj@yahoo.com; nazemi20042003@gmail.com

Abstract. This paper presents the delay fractional optimal control problems (DFOCPs) a modification of the conformable fractional derivative using a novel translation from Caputo-Fabrizio derivative that the kernel is replaced by a suitable exponential function. The delay problem is first transformed to an equivalent problem without delay. By utilizing the necessary optimality conditions and by constructing an error function, an unconstrained minimization problem is defined. A fractional power series neural network for solving the minimization problem is presented. Some illustrative numerical examples are also provided.

1. Introduction

Although, the definition of fractional derivative in [1] has used in many papers and documents, but there are some disadvantages of this fractional definition. Therefore, an efficient improvement of this definition with more simplification is very necessary and meaningful. This is the first novelty of this paper. With help of this new definition of the conformable fractional derivative, we intend to propose a numerical computational approach based on artificial neural network (ANN) scheme for solving a class of DFOCPs.

2. A modification of conformable derivative

The conformable fractional derivative is defined as

$$T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \ t \geq 0. \tag{2.1}$$

Key words and phrases. Time-delay, Fractional optimal control, Fractional power series neural network, Error function.

* Speaker.

Authors in [1] claim that definition (2.1) is the simplest, most natural and efficient definition of fractional derivative. However, one disadvantage of the conformable fractional derivative may be stated by an example:
Let us consider a fractional differential equation by,

$$\begin{cases} T_\alpha f(t) = g(t, x(t)), \ \ t \in [0, a), \ \ 0 < \alpha \leq 1, \\ x(0) = x_0, \\ g(0, x(0)) \neq 0. \end{cases} \tag{2.2}$$

There is no answer for fractional differential equation (2.2) when $t \to 0^+$. According to the definition (2.1), derivative of any function at the point t, when t is very close to zero, is 0 that this is wrong. Therefore, it is necessary to modify the conformable fractional derivative of (2.1).

By using the stated idea in [2], and by replacing $t^{1-\alpha}$ whit $\exp(\frac{1-\alpha}{2-\alpha}t)$, we define a modification of the fractional derivative (2.1) as follows:
Definition 2.4. Given a function $f : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α in (2.1) is modified by

$$\mathcal{T}_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon \exp(\frac{1-\alpha}{2-\alpha}t)) - f(t)}{\varepsilon}, \ t \geq 0, \alpha \in (0, 1]. \tag{2.3}$$

Theorem 2.1. *Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t \geq 0$. Then*

(a) $\mathcal{T}_\alpha(af + bg) = a\mathcal{T}_\alpha(f) + b\mathcal{T}_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
(b) $\mathcal{T}_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
(c) $\mathcal{T}_\alpha(fg) = f\mathcal{T}_\alpha(g) + g\mathcal{T}_\alpha(f)$.
(d) $\mathcal{T}_\alpha(\frac{f}{g}) = \frac{g\mathcal{T}_\alpha(f) - f\mathcal{T}_\alpha(g)}{g^2}$.
(e) *If f is differentiable,* $\mathcal{T}_\alpha(f)(t) = \exp(\frac{1-\alpha}{2-\alpha}t)\frac{df}{dt}(t)$.

3. Problem description and power series neural network

Consider the following problem in which $0 < \alpha \leq 1$,

$$\begin{cases} \text{Minimize } J(x(t), u(t)) = \Phi(t, x(t))|_{t_f} + \int_{t_0}^{t_f} L(t, x(t), x(t-\sigma), u(t), u(t-\tau))dt, \\ \ \textit{subject to} \\ x^\alpha(t) = f_1(t, x(t), x(t-\sigma), u(t), u(t-\tau)), \quad t \in [t_0, t_f], \\ x(t) = \phi(t), \quad t \in [t_0 - \sigma, t_0], \\ u(t) = \psi(t), \quad t \in [t_0 - \tau, t_0], \\ g(t, x(t), u(t)) \leq 0, \quad t \in [t_0, t_f]. \end{cases} \tag{3.1}$$

Using a similar procedure in [3] and Pade approximation, the problem (3.1) is thus transformed to a non-delayed problem

$$\begin{cases} \text{Minimize } J(x(t), u(t)) = \Phi(t, x(t))|_{t_f} + \int_{t_0}^{t_f} L(t, x(t), z(t), u(t), v(t))dt, \\ \ \textit{subjectto} \\ x^\alpha(t) = f_1(t, x(t), z(t), u(t), v(t)), \\ \dot{y}(t) = \frac{4}{\sigma}(x(t) - y(t)) - \dot{x}(t) = f_2(t, x(t), y(t), z(t), u(t), v(t)), \\ \dot{z}(t) = \frac{4}{\sigma}(2y(t) - z(t) - x(t)) + \dot{x}(t) = f_3(t, x(t), y(t), z(t), u(t), v(t)), \\ \dot{w}(t) = \frac{4}{\tau}(u(t) - w(t)) - \dot{u}(t) = f_4(t, w(t), u(t)), \\ \dot{v}(t) = \frac{4}{\tau}(2w(t) - v(t) - u(t)) + \dot{u}(t) = f_5(t, w(t), v(t), u(t), \\ g(t, x(t), u(t)) \leq 0, \quad t \in [t_0, t_f], \\ x(t_0) = \phi(t_0), \ y(t_0) = \phi(t_0 - \frac{\sigma}{2}), \ z(t_0) = \phi(t_0 - \sigma), \\ u(t_0) = \psi(t_0), \ w(t_0) = \psi(t_0 - \frac{\tau}{2}), \ v(t_0) = \psi(t_0 - \tau). \end{cases} \tag{3.2}$$

Now, for estimating solution of the optimality system of the problem (3.2) the trial solutions are written as

$$\begin{cases} x_T = x_0 + (t-t_0)n_x, \\ y_T = y_0 + (t-t_0)n_y, \\ z_T = z_0 + (t-t_0)n_z, \\ w_T = w_0 + (t-t_0)n_w, \\ v_T = v_0 + (t-t_0)n_v, \\ \lambda_T = \dfrac{\partial \Phi}{\partial x}|_{t_f} + (t-t_f)n_\lambda, \\ \gamma_T = (t-t_f)n_\gamma, \\ \mu_T = (t-t_f)n_\mu, \\ \xi_T = (t-t_f)n_\xi, \\ \eta_T = (t-t_f)n_\eta, \\ \zeta_T = (t-t_f)n_\zeta, \\ u_T = n_u, \end{cases} \qquad \begin{cases} n_x = \sum_{i=1}^n w_x^i t^{i\alpha} + b_x^i, \\ n_y = \sum_{i=1}^n w_y^i t^{i\alpha} + b_y^i, \\ n_z = \sum_{i=1}^n w_z^i t^{i\alpha} + b_z^i, \\ n_w = \sum_{i=1}^n w_w^i t^{i\alpha} + b_w^i, \\ n_v = \sum_{i=1}^n w_v^i t^{i\alpha} + b_v^i, \\ n_\lambda = \sum_{i=1}^n w_\lambda^i t^{i\alpha} + b_\lambda^i, \\ n_\gamma = \sum_{i=1}^n w_\gamma^i t^{i\alpha} + b_\gamma^i, \\ n_\mu = \sum_{i=1}^n w_\mu^i t^{i\alpha} + b_\mu^i, \\ n_\xi = \sum_{i=1}^n w_\xi^i t^{i\alpha} + b_\xi^i, \\ n_\eta = \sum_{i=1}^n w_\eta^i t^{i\alpha} + b_\eta^i, \\ n_\zeta = \sum_{i=1}^n w_\zeta^i t^{i\alpha} + b_\zeta^i, \\ n_u = \sum_{i=1}^n w_u^i t^{i\alpha} + b_u^i. \end{cases} \tag{3.3}$$

Approximate solutions should be satisfied in optimality conditions of (3.2). We then get

$$\begin{cases} \lambda_T^\alpha(t) = -\dfrac{\partial H_T}{\partial x_T},\ \dot\gamma_T(t) = -\dfrac{\partial H_T}{\partial y_T},\ \dot\mu_T(t) = -\dfrac{\partial H_T}{\partial z_T},\ \dot\xi_T(t) = -\dfrac{\partial H_T}{\partial w_T},\ \dot\eta_T(t) = -\dfrac{\partial H_T}{\partial v_T},\ x_T^\alpha(t) = \dfrac{\partial H_T}{\partial \lambda_T}, \\ \dot y_T(t) = \dfrac{\partial H_T}{\partial \gamma_T},\ \dot z_T(t) = \dfrac{\partial H_T}{\partial \mu_T},\ \dot w_T(t) = \dfrac{\partial H_T}{\partial \xi_T},\ \dot v_T(t) = \dfrac{\partial H_T}{\partial \eta_T},\ \dfrac{\partial H_T}{\partial u_T} = 0, \\ \phi_{\mathrm{FB}}^\varepsilon(\zeta_T, -g(t, x_T(t), u_T(t))) = 0,\ \varepsilon \to 0_+,\ \dfrac{\partial \Phi}{\partial x_T}|_{t_f} = \lambda_T(t_f), \end{cases} \tag{3.4}$$

where

$$H_T = H(t, x_T, y_T, z_T, w_T, v_T, \lambda_T, \gamma_T, \mu_T, \xi_T, \eta_T, \zeta_T, u_T), \phi_{\mathrm{FB}}^\varepsilon(p,q) = (p+q) - \sqrt{p^2+q^2+\varepsilon},\ \varepsilon \to 0_+.$$

is the Hamiltonian function of We collocate the optimality system (3.4) on the n points t_k, $k = 1, ..., n$ of the interval $[t_0, t_f]$ and then define an optimization problem as

$$\mathrm{minimize}_{\tilde p} E(\tilde p) = \frac{1}{2}\sum_{i=1}^{12}\sum_{k=1}^{n}\{E_i(t_k, \tilde p)\}, \tag{3.5}$$

where

$$\begin{cases} E_1(t_k,\tilde p) = \left[\lambda_T^\alpha(t_k) + \dfrac{\partial H_T}{\partial x_T}\right]^2, E_2(t_k,\tilde p) = \left[\dot\gamma_T(t_k) + \dfrac{\partial H_T}{\partial y_T}\right]^2, E_3(t_k,\tilde p) = \left[\dot\mu_T(t_k) + \dfrac{\partial H_T}{\partial z_T}\right]^2, \\ E_4(t_k,\tilde p) = \left[\dot\xi_T(t_k) + \dfrac{\partial H_T}{\partial w_T}\right]^2, E_5(t_k,\tilde p) = \left[\dot\eta_T(t_k) + \dfrac{\partial H_T}{\partial v_T}\right]^2, E_6(t_k,\tilde p) = \left[x_T^\alpha(t) - \dfrac{\partial H_T}{\partial \lambda_T}\right]^2, \\ E_7(t_k,\tilde p) = \left[\dot y_T(t_k) - \dfrac{\partial H_T}{\partial \gamma_T}\right]^2, E_8(t_k,\tilde p) = \left[\dot z_T(t_k) - \dfrac{\partial H_T}{\partial \mu_T}\right]^2, E_9(t_k,\tilde p) = \left[\dot w_T(t_k) - \dfrac{\partial H_T}{\partial \xi_T}\right]^2, \\ E_{10}(t_k,\tilde p) = \left[\dot v_T(t_k) - \dfrac{\partial H_T}{\partial \eta_T}\right]^2, E_{11}(t_k,\tilde p) = \left[\phi_{\mathrm{FB}}^\varepsilon(\zeta_T, -g(t_k, x_T(t_k), u_T(t_k)))\right]^2, E_{12}(t_k,\tilde p) = \left[\dfrac{\partial H_T}{\partial u_T}\right]^2. \end{cases} \tag{3.6}$$

Example 1.

$$\begin{aligned} \text{Maximize } J = \int_0^{20} e^{-0.05t}(2u(t) - 0.2x(t)^{-1}u(t)^3)dt, \\ x^\alpha(t) = 3x(t)\left(1 - \frac{x(t-0.5)}{5} - u^3(t)\right), \\ x(t) = 2,\ t \in [-0.5, 0],\ 0 < \alpha \le 1, \\ x(t) \ge 2,\ t \in [0, 20], u(t) \ge 0,\ t \in [0, 20]. \end{aligned}$$

The optimal trajectory $x(t)$ and optimal control $u(t)$ are shown in Figures 1 and 2.

FIGURE 1. Optimal state trajectories of $x(t)$ with $\alpha = 1, 0.99, 0.95, 0.89$ for Example 6.3.

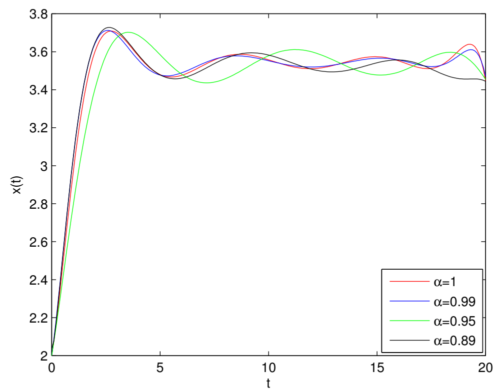
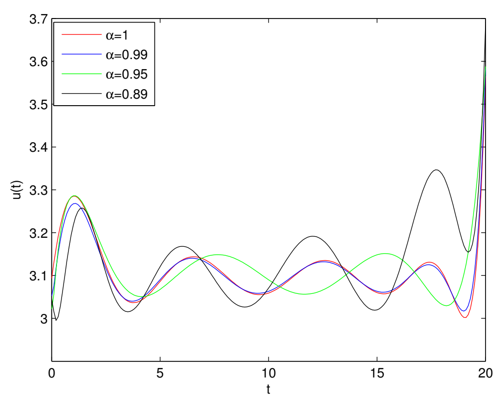


FIGURE 2. Optimal control functions of $u(t)$ with $\alpha = 1, 0.99, 0.95, 0.89$ for Example 6.3.



REFERENCES

1. R. Khalil, M. Al Horani, A. Yousef and M. Sababhehb, A new definition of fractional derivative, Journal of Computational and Applied Mathematics, 264 (2014) 65–70.
2. Caputo M, Fabrizio M . A new definition of fractional derivative without singular kernel. Progr Fract Differ Appl 2015;1:73–85 .
3. S. Hosseinpour, A. R. Nazemi, E. Tohidi, Muntz-Legendre spectral collocation method for solving delay fractional optimal control problems, Journal of Computational and Applied Mathematics, (2019).