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NUMERICAL SOLUTIONS AND SYNCHRONIZATION CONTROL OF FRACTIONAL–ORDER CHAOTIC SYSTEMS WITH NONSTANDARD FINITE DIFFERENCE SCHEMES

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Abstract. In this paper, the numerical solutions of the fractional–order systems are implemented by nonstandard finite difference (NSFD) schemes. In continuation, the synchronization of the fractional–order chaotic systems is studied by using active control technique. The numerical solutions of the master, slave and error systems using the NSFD schemes show that the NSFD approach is easy to implement and accurate.

1. Introduction

In the recent years there is increasing interest in fractional calculus which deals with integration or differentiation of arbitrary orders. The list of applications of fractional calculus has been evergrowing and includes control theory, viscoelasticity, diffusion, turbulence, electromagnetism and many other physical processes. The chaotic synchronization have applied in many different fields since have been discovered, such as biological and physical systems, structural engineering and ecological models. The synchronization of fractional chaotic systems began to attract much attention and it has been raised up some problems. In this paper, we employ the active control technique to study the synchronization between two different chaotic of incommensurate fractional–order systems. The stability theory of linear incommensurate fractional differential equations is utilized to derive the stability of error system which guarantees that the two systems reach complete synchronization, the numerical solutions of the master, slave and error systems using NSFD scheme given by [2] are proposed.

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2. Preliminaries

Derivatives of fractional–order have been introduced in several ways. In this paper we consider Grünwald–Letnikov (GL) approach. The GL method of approximation for the one–dimensional fractional derivative takes the following form

$$D^\alpha x(t) = f(t, x(t)), \qquad x(0) = x_0, \qquad t \in [0, t_f], \tag{2.1}$$

$$D^\alpha x(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[t/h]} (-1)^j \binom{\alpha}{j} x(t - jh),$$

where $0 < \alpha \leq 1$, D^α denotes the fractional derivative, h is the stepsize and $[\frac{t}{h}]$ represents the integer part of $\frac{t}{h}$. Therefore, Eq. (2.1) is discretized in the next form

$$\sum_{j=0}^{n} c_j^\alpha x_{n-j} = f(t_n, x_n), \qquad\qquad n = 1, 2, 3, ...$$

where $t_n = nh$ and c_j^α are the GL coefficients defined as

$$c_j^\alpha = (1 - \frac{1+\alpha}{j}) c_{j-1}^\alpha, \qquad c_0^\alpha = h^{-\alpha}, \qquad j = 1, 2, 3, ...$$

The nonstandard discretization technique is a general scheme where we replace the step-size h by a function $\phi(h)$, see [4, 5]. By applying this technique and using the GL discretization method, it yields the following relations

$$x_{n+1} = c_0^{-\alpha}\Big(- \sum_{j=1}^{n+1} c_j^\alpha x_{n+1-j} + f(t_{n+1}, x_{n+1})\Big), \; n = 0, 1, \ldots$$

where $c_0^\alpha = \phi(h)^{-\alpha}$.

3. Synchronization between Volta's and Lü systems using active control

In this section we study the synchronization between Volta's and Lü systems. Assuming that the Volta's system drives the Lü system, we define the drive (master) and response (slave) systems as follows:

$$\begin{aligned} D^{\alpha_1} x_1 &= -x_1 - a_1 y_1 - z_1 y_1, \\ D^{\alpha_2} y_1 &= -y_1 - b_1 x_1 - x_1 z_1, \\ D^{\alpha_3} z_1 &= c_1 z_1 + x_1 y_1 + 1, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} D^{\alpha_1} x_2 &= a_2 y_2 - a_2 x_2 + u_1, \\ D^{\alpha_2} y_2 &= c_2 y_2 - x_2 z_2 + u_2, \\ D^{\alpha_3} z_2 &= x_2 y_2 - b_2 z_2 + u_3. \end{aligned} \tag{3.2}$$

The unknown terms u_1, u_2, u_3 in (3.2) are active control functions to be determined. Define the error functions as

$$e_1 = x_2 - x_1, \qquad e_2 = y_2 - y_1, \qquad e_3 = z_2 - z_1. \tag{3.3}$$

Equation (3.3) together with (3.1) and (3.2) yields the error system

$$\begin{aligned}
D^{\alpha_1}e_1 &= -a_2e_1 + (1-a_2)x_1 + a_2y_2 + a_1y_1 + z_1y_1 + u_1,\\
D^{\alpha_2}e_2 &= c_2e_2 + (1+c_2)y_1 - x_2z_2 + b_1x_1 + x_1z_1 + u_2,\\
D^{\alpha_3}e_3 &= -b_2e_3 - (b_2+c_1)z_1 + x_2y_2 - x_1y_1 - 1 + u_3.
\end{aligned} \tag{3.4}$$

We define active control functions u_i as

$$\begin{aligned}
u_1 &= V_1 - (1-a_2)x_1 - a_2y_2 - a_1y_1 - z_1y_1,\\
u_2 &= V_2 - (1+c_2)y_1 + x_2z_2 - b_1x_1 - x_1z_1,\\
u_3 &= V_3 + (b_2+c_1)z_1 - x_2y_2 + x_1y_1 + 1,
\end{aligned} \tag{3.5}$$

where $V_1(t)$, $V_2(t)$ and $V_3(t)$ are the control inputs. Substituting (3.5) into (3.4) gives

$$\begin{aligned}
D^{\alpha_1}e_1 &= -a_2e_1 + V_1(t),\\
D^{\alpha_2}e_2 &= c_2e_2 + V_2(t),\\
D^{\alpha_3}e_3 &= -b_2e_3 + V_3(t).
\end{aligned} \tag{3.6}$$

The error system (3.6) is a linear system with the control input function $[V_1\ V_2\ V_3]^T = H[e_1\ e_2\ e_1]^T$, where H is a real matrix, chosen so that for all eigenvalues λ_i of the system (3.6) satisfy: $|\arg(\lambda_i)| > \dfrac{\alpha\pi}{2}$, $i = 1,2,3$, we choose

$$H = \begin{pmatrix} -1+a_2 & 0 & 0\\ 0 & -1-c_2 & 0\\ 0 & 0 & -1+b_2 \end{pmatrix}.$$

All the three eigenvalues of the system (3.6) are -1; hence the condition that all $\alpha_i \le 1$, $i = 1,2,3$ is satisfied, and we finally obtain the required synchronization.

3.1. **Simulation and Results.** If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ then the Volta's system is called commensurate otherwise incommensurate, a minimal order α for chaotic behavior can be determinned [3] and it is $\alpha > 0.99$. This system is equivalent to the classical integer–order Volta's system when $\alpha = 1$, which is chaotic at $(a_1, b_1, c_1) = (5, 85, 0.5)$ and the uncontrolled system Lü displays chaotic behavior when $\alpha > 0.91$ and $(a_2, b_2, c_2) = (36, 3, 20)$, see [1].

In Figure 1 is depicted phase trajectory of the fractional–order Volta's system for commensurate order $\alpha_1 = \alpha_2 = \alpha_3 = 1$ with parameters $(a_1, b_1, c_1) = (5, 85, 0.5)$ with the initial conditions $(x_0, y_0, z_0) = (8, 2, 1)$, for simulation time 20s and stepsize $h = 0.001$.

Figure 2 shows the chaotic behavior for fractional–order Lü system, where system parameters are $(a_2, b_2, c_2) = (36, 3, 20)$, incommensurate orders of the derivatives are $\alpha_1 = 0.93$, $\alpha_2 = 0.99$ and $\alpha_3 = 0.98$ and the initial conditions are $(x_0, y_0, z_0) = (0.2, 0.5, 0.3)$ for the simulation time $t = 20$s and stepsize $h = 0.001$.

Figure 3 shows the synchronized states and synchronization errors for $\alpha = 0.98$.

Conclusions

In the present article it is shown that the incommensurate different fractional–order chaotic systems can be synchronized using active control. The incommensurate fractional–order Volta's system is synchronized with incommensurate fractional Lü system. Further

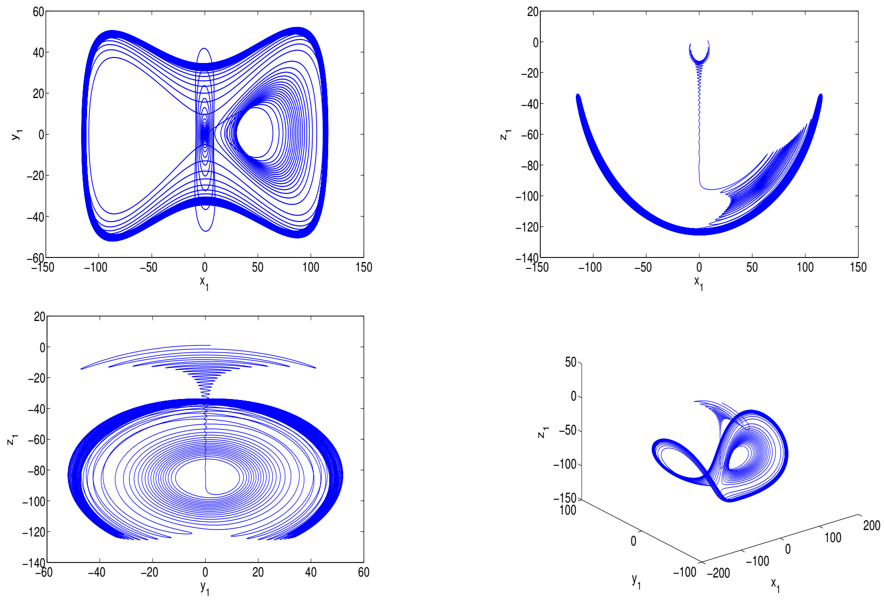

FIGURE 1. Chaotic attractor of the Volta's system projected into 2D phase planes and 3D state space for derivative orders $\alpha = 1$ and $h = 0.001$.

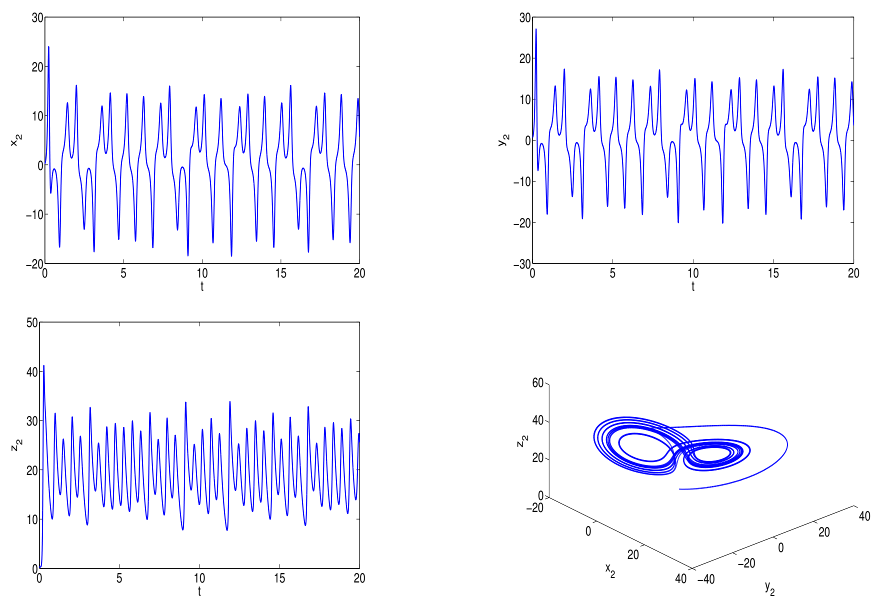

FIGURE 2. Chaotic attractor of the uncontrolled system Lü projected into 2D phase planes and 3D state space for derivative orders $\alpha_1 = 0.93$, $\alpha_2 = 0.99$, $\alpha_3 = 0.98$ and $h = 0.001$.

it is observed from the Figures that the synchronization starts earlier for higher values of system order.

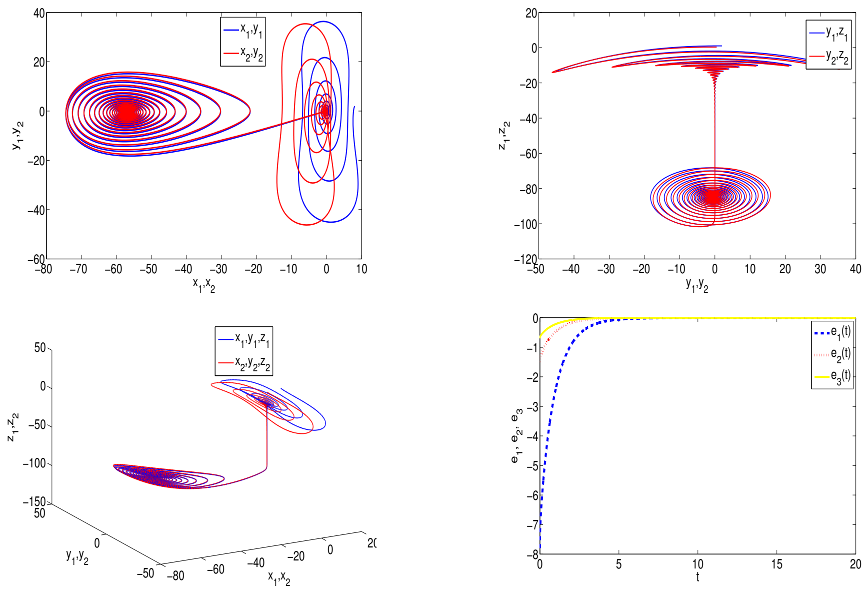

Figure 3. Chaotic attractor of Volta's and Lü systems projected into 2D phase planes and the error functions both systems for derivative orders $\alpha = 0.98$ and $h = 0.001$.

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